Analytical solutions for the preliminary estimation of long-term rates of groundwater inflow into excavations:
Long excavations and circular excavations

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Overview

A set of steady-state analytical solutions of groundwater inflows to open excavations is assembled. The solutions are appropriate for developing preliminary estimates of long-term rates of groundwater flows into open excavations.

The solutions incorporate the following assumptions:

- The aquifer is a continuous porous medium;
- The aquifer is homogeneous and isotropic; and
- Flow is steady and laminar.

Ten solutions are presented for two cases: flow into the sides of a long excavation (linear flow) and flow into the sides of a circular excavation (radial flow).

Solutions for five conceptual models are provided for each of these two cases:

- Flow through a confined aquifer;
- Flow through an unconfined flow without recharge;
- Combined confined/unconfined flow;
- Flow through an unconfined flow with recharge; and
- Flow through an aquifer that is overlain by a leaky aquitard.

Two additional solutions are presented for the estimation of the flow into the base of a circular excavation.

References for the solutions are provided. For completeness, the derivations of the solutions are included in appendices.

A separate report has been prepared to summarize approaches for estimating inflows to rectangular excavations.

Contents

Part 1: Steady-state flow into the sides of a long excavation

1. Linear flow into the sides of an excavation in a confined aquifer

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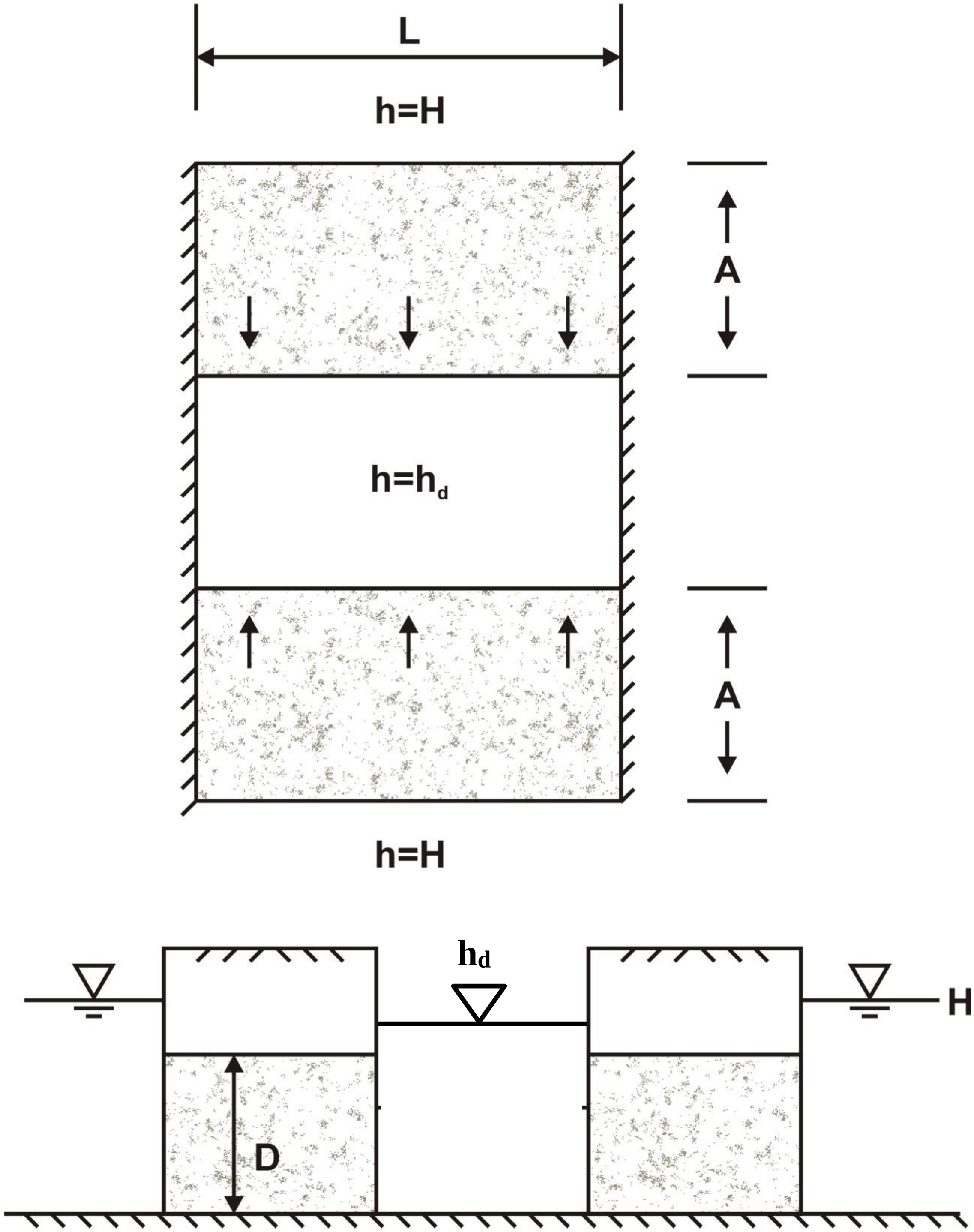
3. Linear flow into the sides of an excavation in an aquifer with conversion between unconfined and confined conditions

4. Linear flow into the sides of an excavation in an unconfined aquifer with recharge

5. Linear flow into the sides of an excavation in a confined aquifer that is overlain by a leaky aquitard

1. Model 1: Linear flow into the sides of an excavation in a confined aquifer



The inflow into both sides of an excavation of length L is:

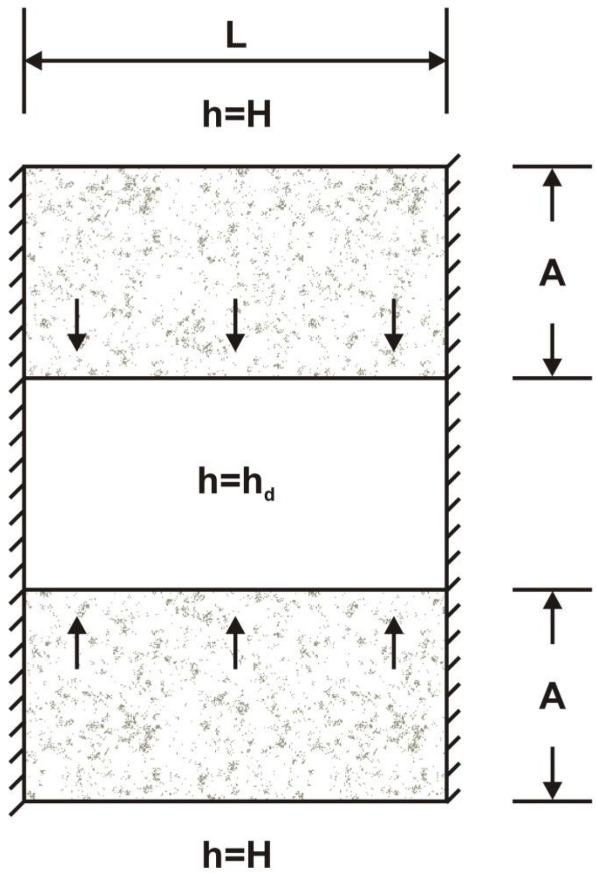
$$\boxed{Q = -2KD\frac{(H-h_d)}{A}\,L}$$

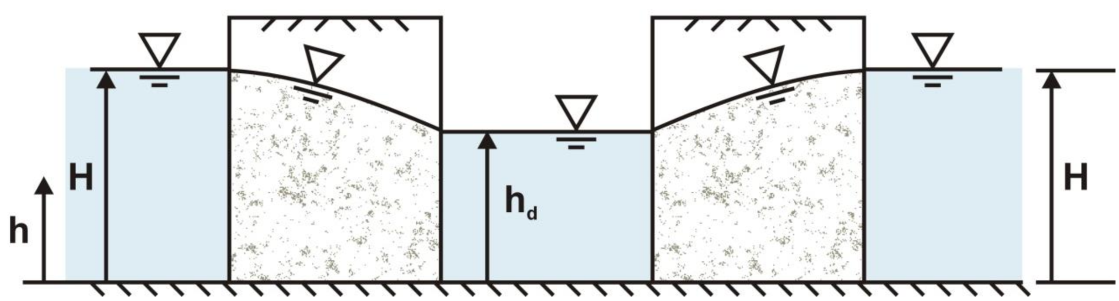
The negative sign denotes flow out of the aquifer into the excavation for $h_d < H$.

References:

- The solution is presented in Mansur and Kaufman (1962; Equation [3-6]).
- The derivation of the solution is included in Appendix A.

2. Model 2: Linear flow into the sides of an excavation in an unconfined aquifer





The inflow into both sides of an excavation of length L is:

$$\boxed{Q = -K\frac{(H^2 - h_d^2)}{A}\,L}$$

The heads H and h_d are measured with respect to the base of the aquifer.
The negative sign denotes flow out of the aquifer into the excavation for $h_d < H$.

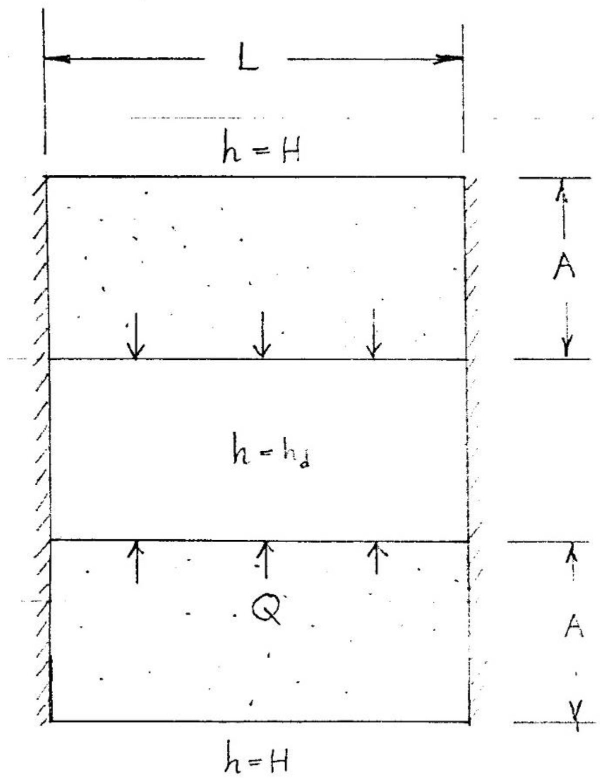
References:

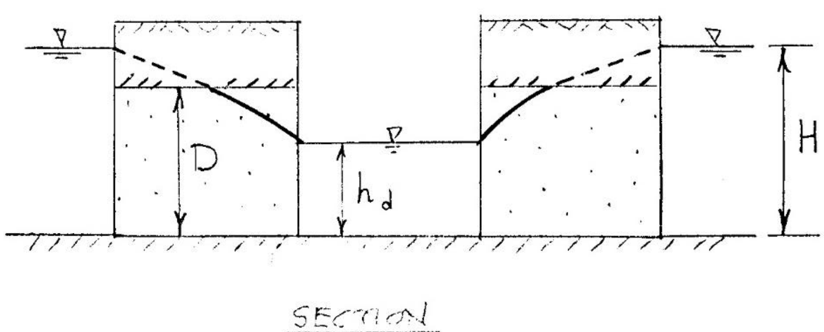
- The solution is presented in Mansur and Kaufman (1962; Equation [3-11]) and is a special case of Bear (1979; Equation [5-213]) for no recharge [$N = 0$].

- The derivation of the solution is included in Appendix A. The solution for the head is derived with the Dupuit-Forchheimer approximation but the solution for the discharge is exact.

3. Model 3: Linear flow into the sides of an excavation in an aquifer with conversion between unconfined and confined conditions

The water level in the excavation is lowered below the top of the aquifer.





The inflow into <u>both</u> sides of an excavation of length *L* is:

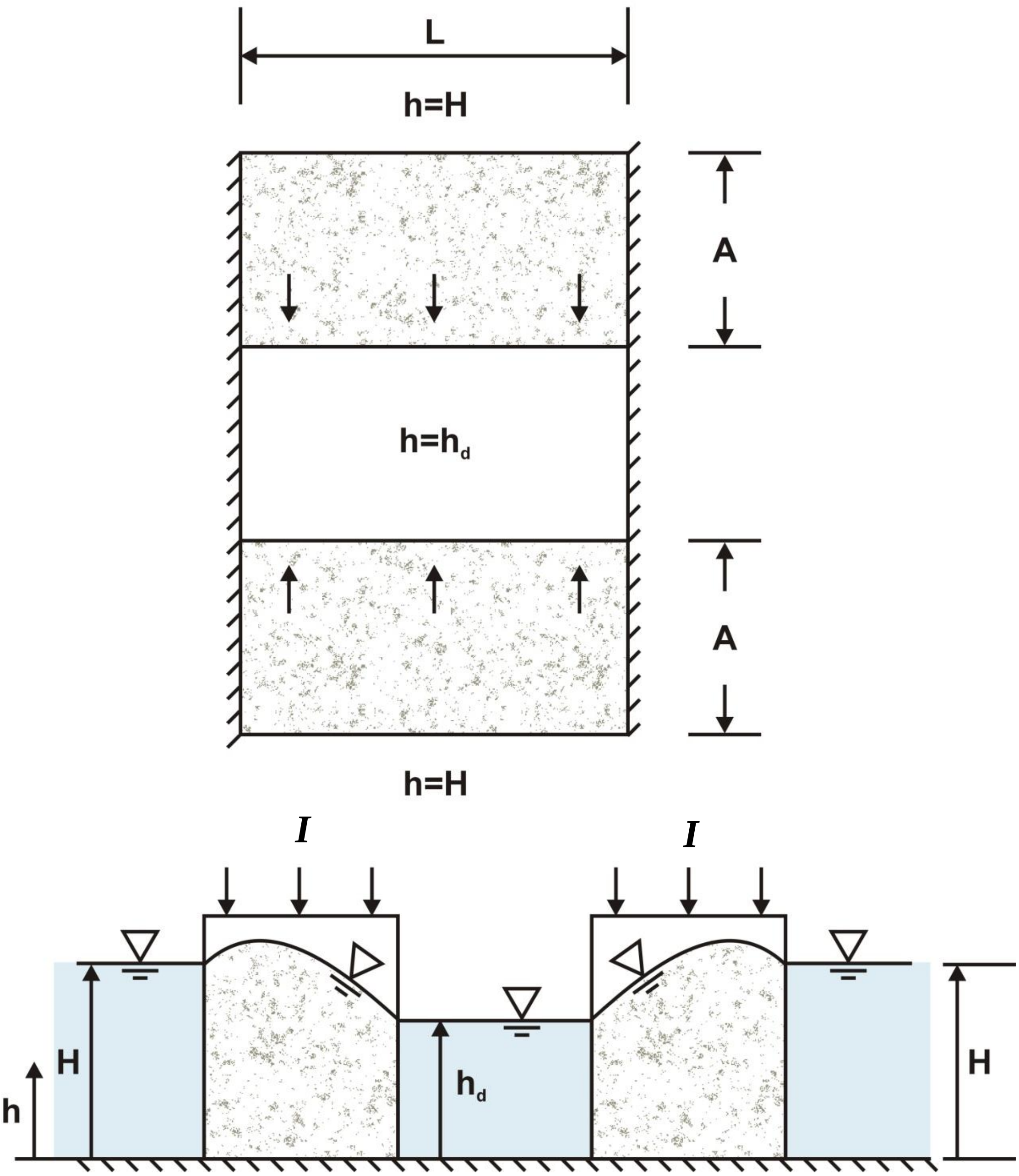
$$\boxed{Q = -K\frac{(2DH - D^2 - h_d^2)}{A}\,L}$$

The heads H and h_d are measured with respect to the base of the aquifer.
The negative sign denotes flow out of the aquifer into the excavation for $h_d < H$.

References:

- The solution is presented in Mansur and Kaufman (1962; Equation [3-18]).
- The derivation of the solution is included in Appendix A.

4. Model 4: Linear flow into the sides of an excavation in an unconfined aquifer with recharge



For steady recharge at a rate *I*, the discharge into both sides the excavation of length *L* is:

$$\boxed{Q = -K\left[\frac{(H^2 - h_d^2)}{A} + \frac{IA}{K}\right] L}$$

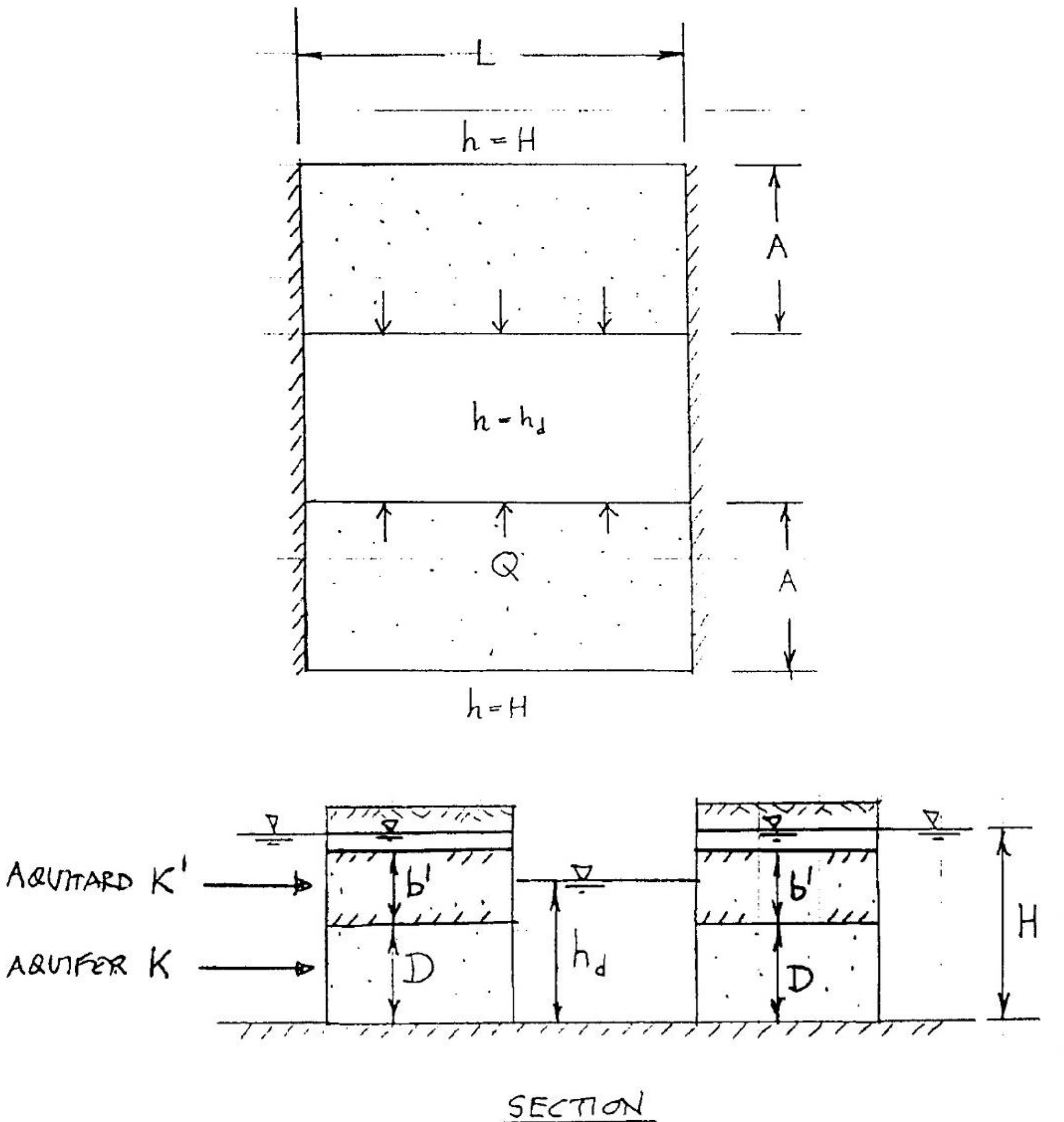
The heads *H* and h_d are measured with respect to the base of the aquifer.
The negative sign denotes flow out of the aquifer into the excavation for $h_d < H$.

References:

- The solution is presented in Bear (1979; Equation [5-213]).

- The derivation of the solution is included in Appendix A. The solution for the head is derived with the Dupuit-Forchheimer approximation but the solution for the discharge is exact.

5. **Model 5: Linear flow into the sides of an excavation in a confined aquifer that is overlain by a leaky aquitard**



The inflow into <u>both</u> sides of an excavation of length *L* is:

$$Q = -2\frac{KD}{\lambda}(H - h_d)\frac{\left(1 + EXP\left\{\frac{-2A}{\lambda}\right\}\right)}{\left(1 - EXP\left\{\frac{-2A}{\lambda}\right\}\right)}\,L$$

$$\lambda = \left[\frac{KD}{(K'/b')}\right]^{1/2}$$

The negative sign denotes flow out of the aquifer into the excavation for $h_d < H$.

<u>Reference</u>:

The solution is presented her for the first time. The derivation of the solution is presented in Appendix A. The derivation follows approaches of Huisman (1972). The solution for the special case of an aquifer that is semi-infinite in length is given in Bear (1979; Equation [5-29]).

Part 2: Steady-state radial flow into the sides of a circular excavation

6. Radial flow into the sides of a circular excavation in a confined aquifer

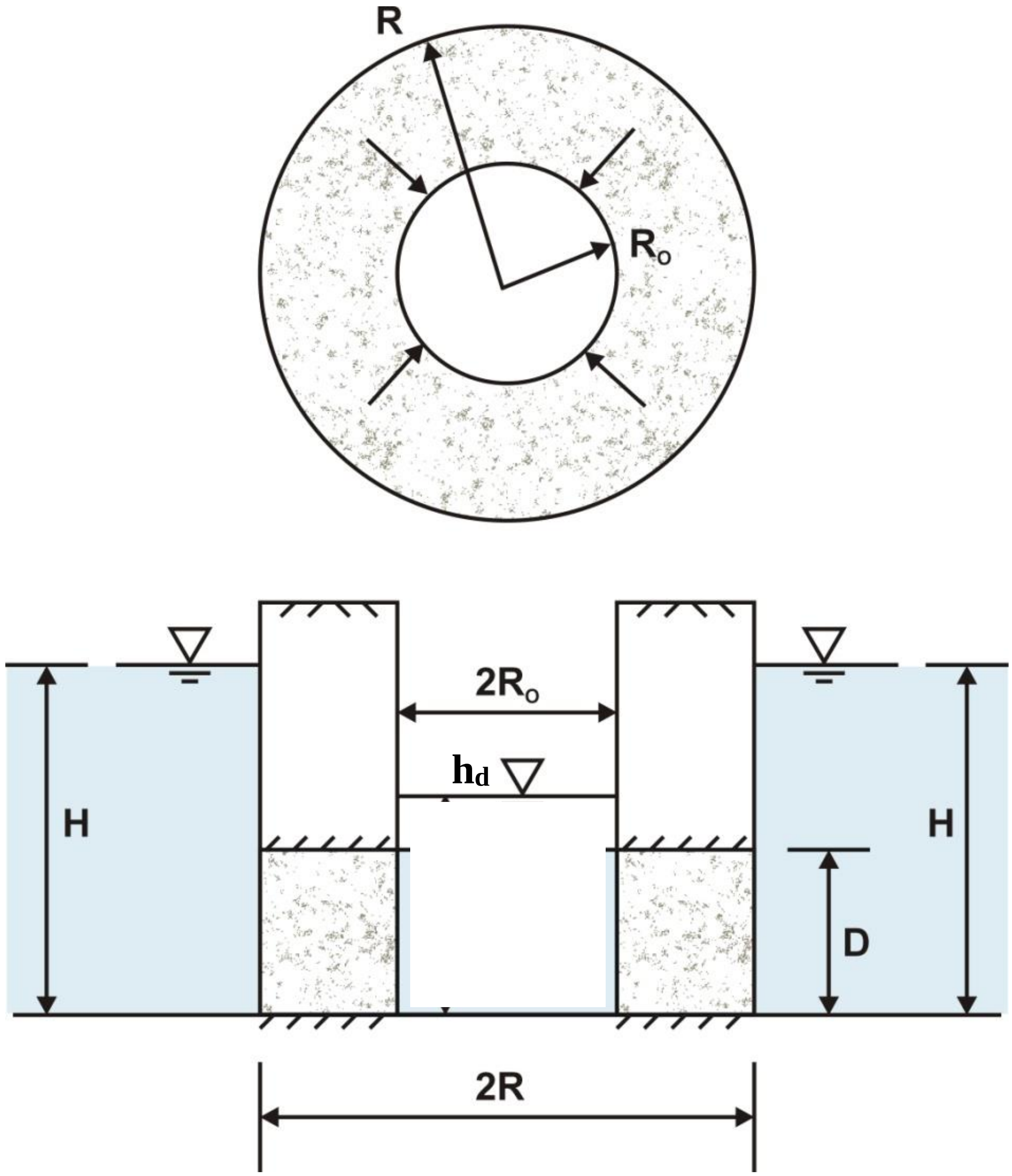
7. Radial flow into the sides of a circular excavation in an unconfined aquifer

8. Radial flow into the sides of a circular excavation in an aquifer with conversion between unconfined and confined conditions

9. Radial flow into the sides of a circular excavation in an unconfined aquifer with recharge

10. Radial flow into the sides of a circular excavation in a confined aquifer that is overlain by a leaky aquitard

6. Model 6: Radial flow into the sides of a circular excavation in a confined aquifer



The inflow into the excavation is:

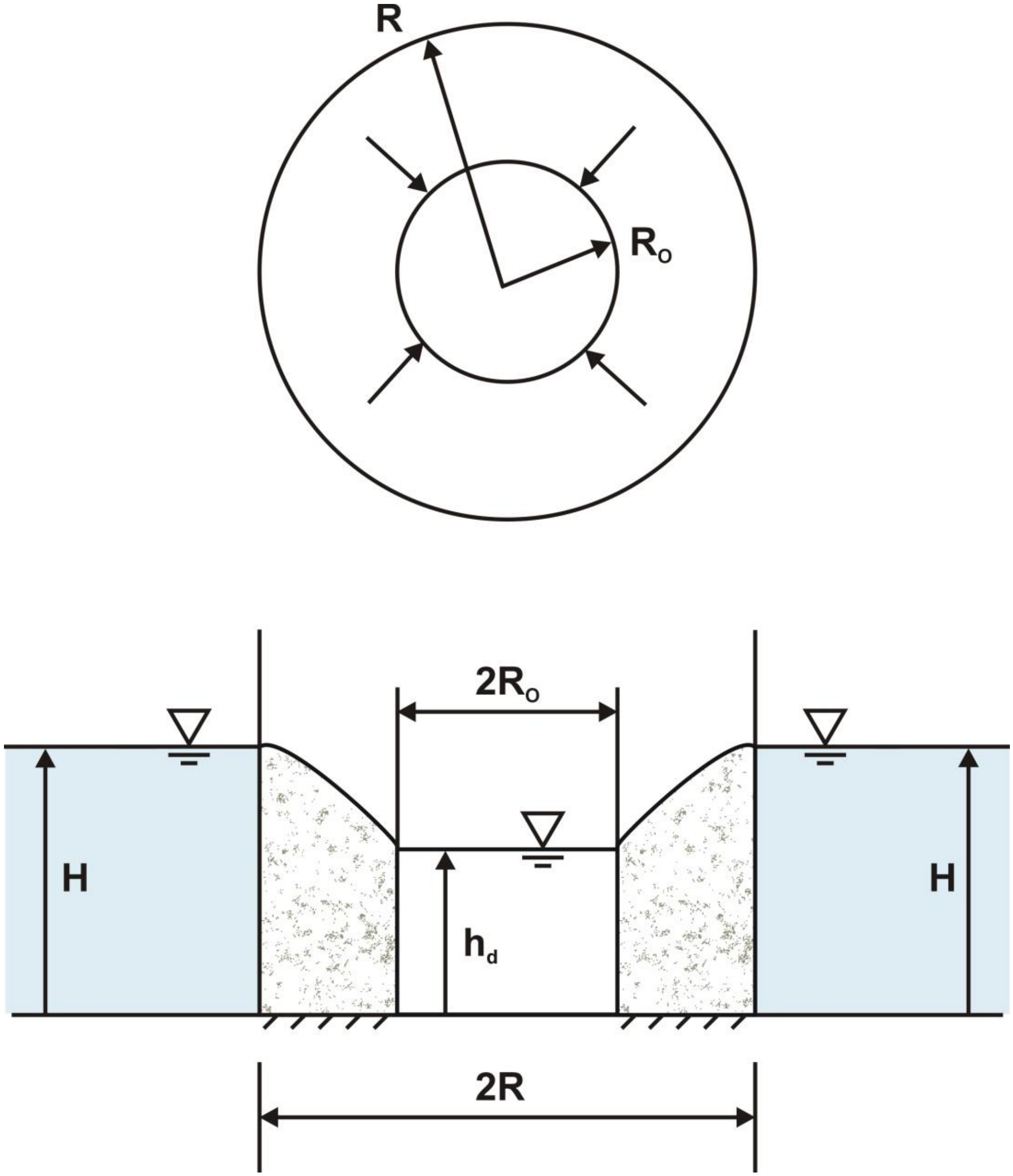
$$\boxed{Q = -2\pi KD \frac{(H - h_d)}{\ln\left\{\frac{R}{R_0}\right\}}}$$

The negative sign denotes flow out of the aquifer into the excavation for $h_d < H$.

<u>References</u>:

- The solution is referred to as the *Thiem solution* and is presented in Mansur and Kaufman (1962; Equation [3-47]).
- The derivation of the solution is included in Appendix B.

7. Model 7: Radial flow into the sides of a circular excavation in an unconfined aquifer



The inflow into the excavation is:

$$\boxed{Q = -\pi K \frac{(H^2 - h_d^2)}{\ln\left\{\frac{R}{R_0}\right\}}}$$

The heads H and h_d are measured with respect to the base of the aquifer.
The negative sign denotes flow out of the aquifer into the excavation for $h_d < H$.

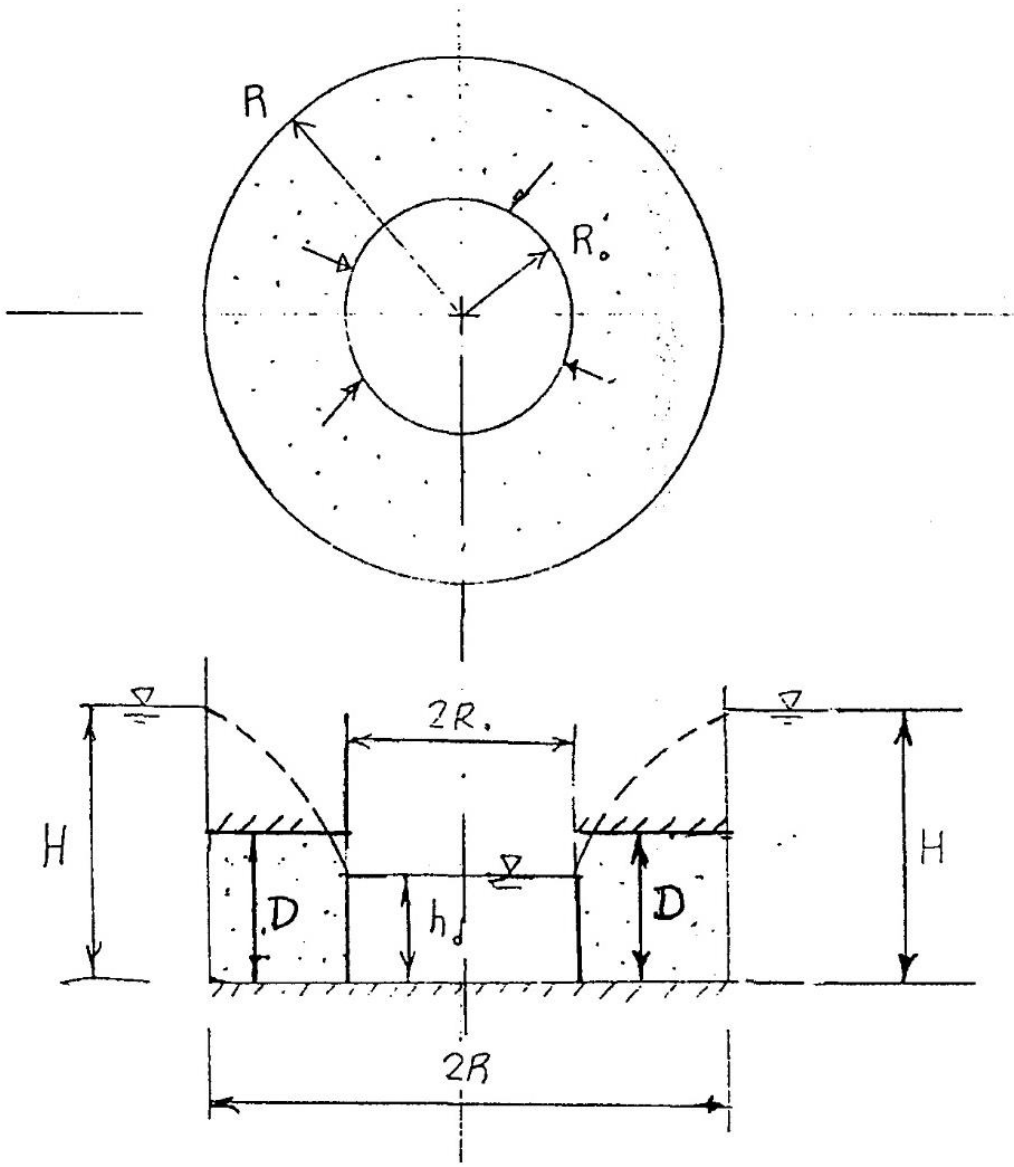
<u>References</u>:

- The solution is referred to as the *Dupuit solution* and is presented in Mansur and Kaufman (1962; Equation [3-57]) and Bear (1979; Equation [8-24]).

- The derivation of the solution is included in Appendix B. The solution for the head profile is derived with the Dupuit-Forchheimer approximation, but the solution for the discharge is exact.

8. Model 8: Radial flow into the sides of a circular excavation in an aquifer with conversion between unconfined and confined conditions

The water level in the excavation is lowered below the top of the aquifer.



The inflow into the excavation is:

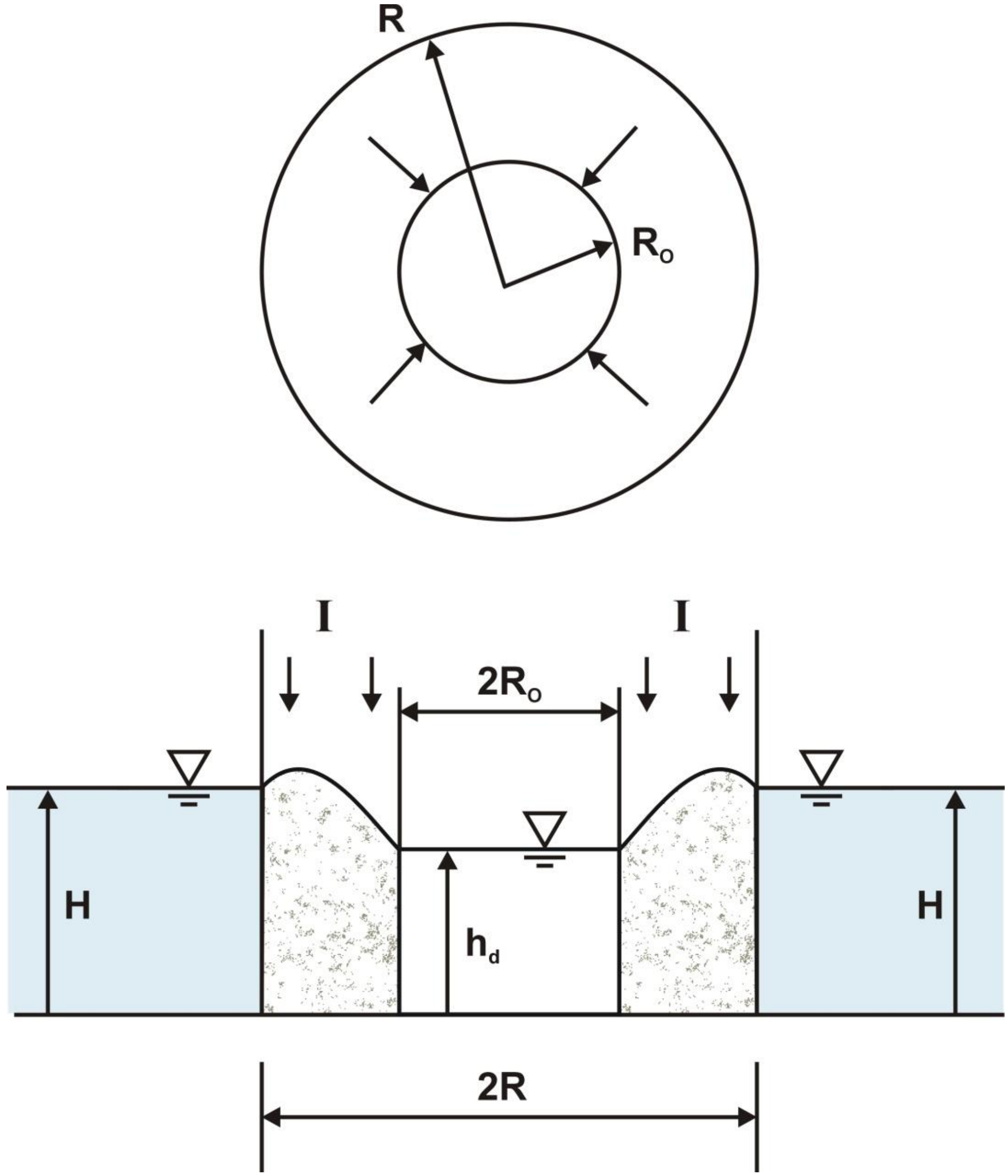
$$\boxed{Q = -\pi K \frac{(2DH - D^2 - h_d^2)}{\ln\left\{\frac{R}{R_0}\right\}}}$$

The heads H and h_d are measured with respect to the base of the aquifer.
The negative sign denotes flow out of the aquifer into the excavation for $h_d < H$.

References:

- This solution is presented in Mansur and Kaufman (1962; Equation [3-67]).
- The derivation of the solution is included in Appendix B.

9. Model 9: Radial flow into the sides of a circular excavation in an unconfined aquifer with recharge



For steady recharge at a rate I, the inflow into the excavation is:

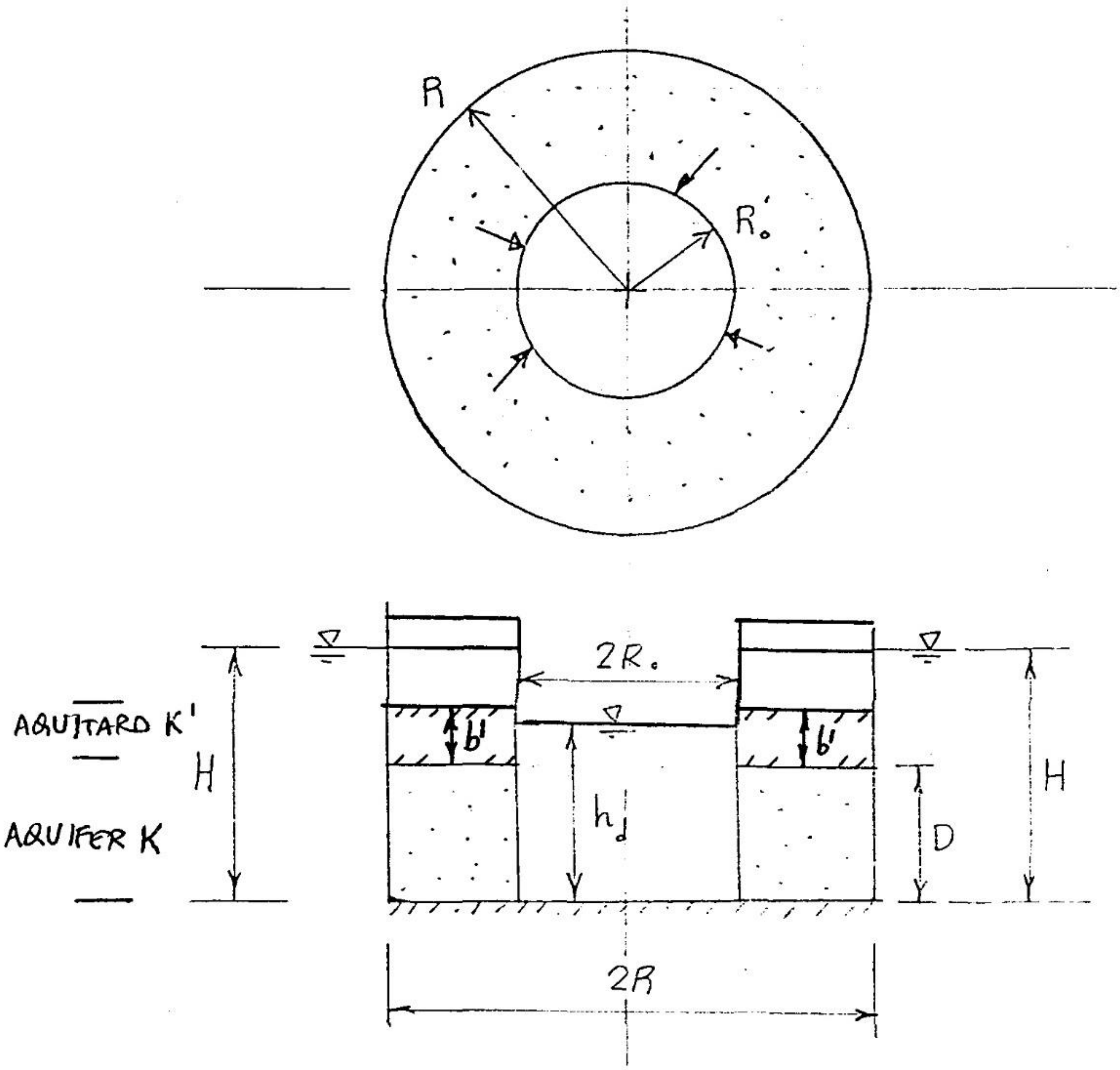
$$Q = -\frac{\pi K}{ln\left\{\frac{R}{R_0}\right\}}\left[(H^2 - h_d^2) + \frac{I}{2K}(R^2 - R_0^2) - \frac{IR_0^2}{K} ln\left\{\frac{R}{R_0}\right\}\right]$$

The heads H and h_d are measured with respect to the base of the aquifer.
The negative sign denotes flow out of the aquifer into the excavation for $h_d < H$.

References:

- This solution for the discharge is obtained by re-arranging Equation [8-34]) in Bear (1979).
- The derivation of the solution is included in Appendix B. The solution for the head profile is derived with the Dupuit-Forchheimer approximation, but the solution for the discharge is exact.

10. Model 10: Radial flow into the sides of a circular excavation in a confined aquifer that is overlain by a leaky aquitard



The inflow into the excavation is:

$$Q = 2\pi KD \frac{R_0}{\lambda}(H - h_d) \frac{\left[I_1\left(\frac{R_0}{\lambda}\right) K_0\left(\frac{R}{\lambda}\right) + I_0\left(\frac{R}{\lambda}\right) K_1\left(\frac{R_0}{\lambda}\right)\right]}{\left[I_0\left(\frac{R_0}{\lambda}\right) K_0\left(\frac{R}{\lambda}\right) - I_0\left(\frac{R}{\lambda}\right) K_0\left(\frac{R_0}{\lambda}\right)\right]}$$

$$\lambda = \left[\frac{KD}{(K'/b')}\right]^{1/2}$$

The functions I_0, I_1, K_0 and K_1 are defined as follows:

I_0 Modified Bessel function of the first kind, order 0
I_1 Modified Bessel function of the first kind, order 1
K_0 Modified Bessel function of the second kind, order 0
K_1 Modified Bessel function of the second kind, order 1

The denominator in the expression for Q is negative; therefore, Q is negative for $h_d < H$.
The negative sign denotes flow out of the aquifer into the excavation.

References:

The solution is presented her for the first time. The derivation of the solution is presented in Appendix B, following the general approach of Huisman (1972) and Bear (1979; Section 8-4).

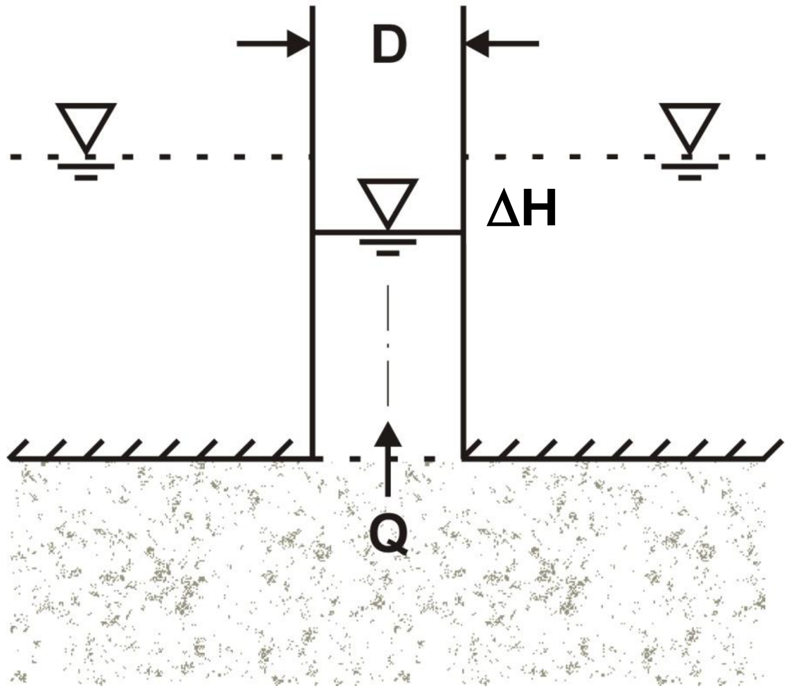
Part 3: Steady-state flow into the base of a circular excavation

11. Forchheimer (1914) solution

12. Hvorslev (1951) Case 4/C

11. Model 11: Flow into the base of a circular excavation: Forchheimer (1914) solution

The conceptual model for the Forchheimer (1914) solution [also Hvorslev (1951) Case 2] is illustrated below. The circular excavation of diameter *D* is open to a confined aquifer only across its bottom. Applications of the solution are presented in Suzuki and Yokoya (1992) and Marinelli and Niccoli (2000).



Conceptual model for the Forchheimer (1914) solution

The Forchheimer (1914) solution for the flow rate into the bottom of the excavation is:

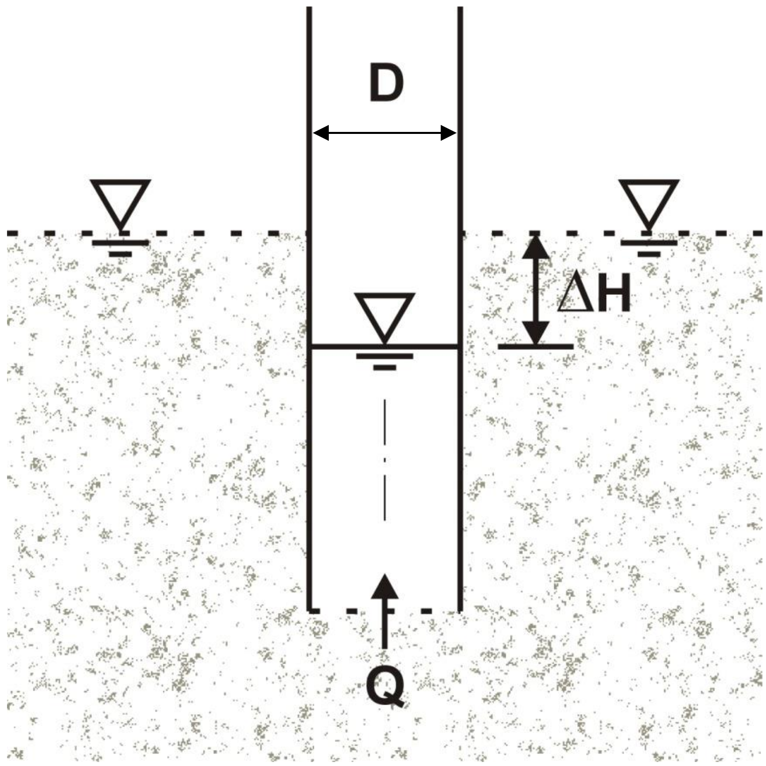
$$Q = 2D * K\Delta H$$

In terms of the radius of the circular excavation, R_0, the solution is written as:

$$\boxed{Q = 4R_0 * K\Delta H}$$

12. Model 12: Flow into the base of a circular excavation: Hvorslev (1951) Case 4/C

The inflow the base of a circular excavation in an extensive formation has been analyzed by Harza (1935) and Taylor (1948). The results of their analyses are reproduced as Hvorslev (1951) Case 4/C. The conceptual model for this case is illustrated below. Silvestri et al. (2012) have derived an exact solution that has this problem as a limiting case. The results of their analysis are nearly identical to those of Harza and Taylor (Neville, 2013).



Conceptual model for Hvorslev (1951) Case 4/C

The flow rate into the bottom of the excavation is approximately:

$$Q = 2.75D * K\Delta H$$

In terms of the radius of the circular excavation, R_0, the solution is written as:

$$\boxed{Q = 5.5R_0 * K\Delta H}$$

References

Bear, J., 1979: **Hydraulics of Groundwater**, McGraw-Hill Inc., New York, New York.

Forchheimer, P., 1914: **Hydraulik**, B.G. Teubner, Leipzig and Berlin, p. 439

Harza, L.F., 1935: Uplift and seepage under dams, *Transactions of the American Society of Civil Engineers*, vol. 100, pp. 1352-1385.

Huisman, L., 1972: **Groundwater Recovery**, Winchester Press, New York, New York.

Hvorslev, M.J., 1951: Time Lag and Soil Permeability in Ground-Water Observations, Bulletin No. 36, Waterways Experiment Station, U.S. Army Corps of Engineers, Vicksburg, Mississippi, 50 p.

Mansur, C.I., and R.I. Kaufman, 1962: *Dewatering*, in **Foundation Engineering**, G.A. Leonards (ed.), McGraw-Hill Inc., New York, New York.

Marinelli, F., and W.L. Niccoli, 2000: Simple analytical equations for estimating ground water inflow to a mine pit, *Ground Water*, vol. 38, no. 2, pp. 311 314.

Neville, C.J., 2013: discussion of "Shape factors for cylindrical piezometers in uniform soil", *Ground Water*, vol. 51, no. 2, pp. 168-169.

Reddi, L.N., 2003: **Seepage in Soils**, John Wiley & Sons, Hoboken, New Jersey.

Silvestri, V., G. Abou-Samra, and C. Bravo-Jonard, 2012: Shape factors for cylindrical piezometers in uniform soil, *Ground Water*, vol. 50, no. 2, pp. 279-284.

Suzuki, O., and H. Yokoya, 1992: Application of Forchheimer's formula to dewatered excavation as a large circular well, *Soils and Foundations*, vol. 32, no. 1, pp. 215-221.

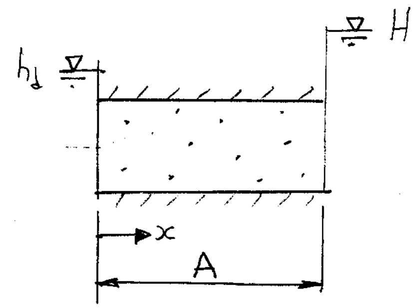
Taylor, D.W., 1948: **Fundamentals of Soil Mechanics**, John Wiley & Sons, New York, New York.

APPENDIX A

ANALYTICAL SOLUTION FOR LINEAR CONFINED FLOW INTO AN EXCAVATION

1. HEAD SOLUTION

$$\frac{d}{dx}\left(KD\frac{dh}{dx}\right) = 0 \qquad 0 \le x \le A$$



SUBJECT TO :

$$h(0) = h_d$$
$$h(A) = H$$

Integrating the governing equation twice yields:

$$h = \frac{1}{KD} C_1 x + C_2$$

where C_1 and C_2 are as-yet-undetermined constants of integration.

The coefficients are determined by evaluating the boundary conditions:

i) $h(0) = h_D = C_2$

ii) $h(A) = H = \frac{1}{KD} C_1 A + h_D$

$$\rightarrow C_1 = (H - h_d) \frac{KD}{A}$$

$$\therefore \quad h = \frac{1}{KD} \left((H - h_d) \frac{KD}{A} \right) x + h_D$$

Simplifying:

$$\boxed{h = h_D + (H - h_d) \frac{x}{A}}$$

2. SOLUTION FOR DISCHARGE

For an excavation of length L, the flow into one face of the excavation is given by:

$$Q = -K \left.\frac{dh}{dx}\right|_{x=0} \cdot DL$$

now $\frac{dh}{dx} = \frac{(H-h_d)}{A}$ [The gradient is uniform.]

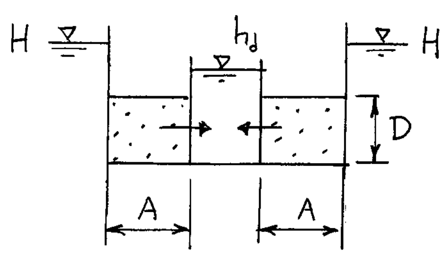
$$\therefore \boxed{Q = -KD\frac{(H-h_d)}{A}L}$$

CHECK:

Reddi (2003; p. 106)

→ This is the two one side of an excavation.

For a symmetric situation the actual flow is doubled.

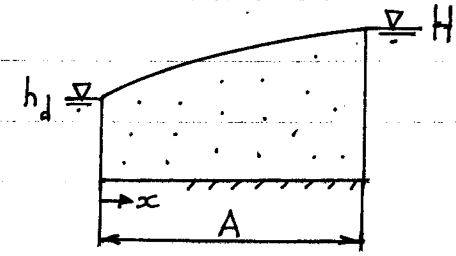


MODEL 2 DERIVATION

ANALYTICAL SOLUTION FOR UNCONFINED LINEAR FLOW INTO AN EXCAVATION

1. HEAD SOLUTION

$$\frac{d}{dx}\left(K h \frac{dh}{dx}\right) = 0 \quad ; \; 0 \le x \le A$$



SUBJECT TO : $h(0) = h_d$

$h(A) = H$

Apply a transformation, define : $u = h^2$

$\therefore \; h = u^{1/2}$

$dh = \frac{1}{2} u^{-1/2} \, du$

substituting into the governing equation :

$$\frac{d}{dx}\left(K \, u^{1/2} \cdot \frac{1}{2} u^{-1/2} \frac{du}{dx}\right) = 0$$

Simplifying :

$$\frac{d}{dx}\left(\frac{K}{2} \frac{du}{dx}\right) = 0$$

The transformed boundary conditions are :

$u(0) = h_d^2$

$u(A) = H^2$

Integrating twice :

$$u = \frac{2}{K} C_1 x + C_2$$

Evaluating the boundary conditions :

i) $u(0) = h_d^2 = C_2$

ii) $u(A) = H^2 = \frac{2}{K} C_1 A + h_d^2$

$$\rightarrow C_1 = (H^2 - h_d^2) \frac{K}{2A}$$

$$\therefore \quad u = \frac{2}{K}\left((H^2 - h_d^2) \frac{K}{2A} \right) x + h_d^2$$

Simplifying :

$$u = h_d^2 + (H^2 - h_d^2) \frac{x}{A}$$

$$\therefore \quad \boxed{h = \left[h_d^2 + (H^2 - h_d^2) \frac{x}{A} \right]^{1/2}}$$

2. SOLUTION FOR DISCHARGE

The flow to one face of the excavation of length L is given by:

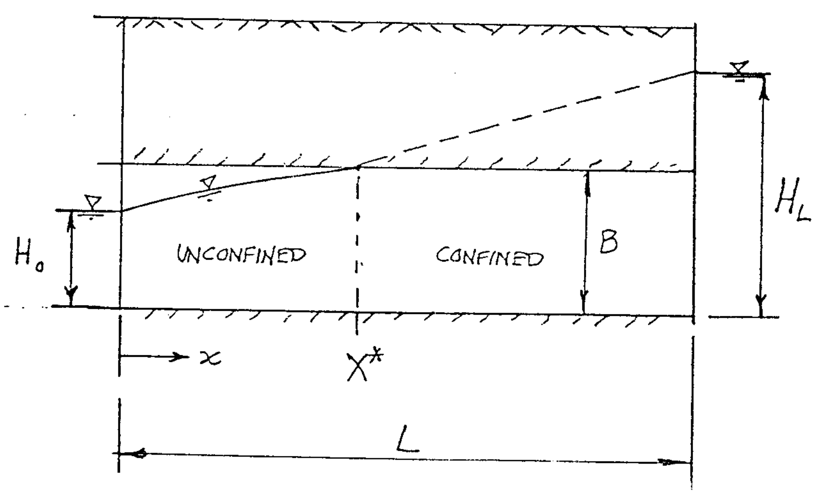
$$Q = -K \cdot \left.\frac{dh}{dx}\right|_{x=0} \left. h \right|_{x=0} L$$

$$= -K \left.\frac{1}{2}\frac{du}{dx}\right|_{x=0} L$$

now $\dfrac{du}{dx} = \dfrac{(H^2 - h_d^2)}{A}$

$$\therefore \boxed{Q = -\frac{K}{2}\,\frac{(H^2 - h_d^2)}{A}\,L}$$

STEADY LINEAR FLOW WITH CONVERSION FROM CONFINED TO UNCONFINED CONDITIONS



The conversion from unconfined to confined conditions occurs at a distance X^* from the unconfined boundary.

h is measured wrt the base of the aquifer.

1. SOLUTION FOR LOCATION OF X^*

i) $0 \le x \le X^*$: UNCONFINED FLOW

$$\frac{Q}{W} = \frac{K}{2}\frac{(B^2 - H_0^2)}{X^*} \quad \text{——(1)}$$

ii) $X^* \le x \le L$: CONFINED FLOW

$$\frac{Q}{W} = KB\frac{(H_L - B)}{(L - X^*)} \quad \text{——(2)}$$

SOLVE FOR X^* BY EQUATING (1) AND (2):

$$\frac{K}{2}\frac{(B^2 - H_0^2)}{X^*} = KB\frac{(H_L - B)}{(L - X^*)}$$

EXPANDING :

$$(L - X^*)(B^2 - H_0^2) = 2BX^*(H_L - B)$$

$$\rightarrow \quad LB^2 - LH_o^2 - X^*B^2 + X^*H_o^2 = 2BX^*H_L - 2BX^*B$$

COLLECTING TERMS :

$$LB^2 - LH_o^2 = 2BX^*H_L - 2BX^*B + X^*B^2 - X^*H_o^2$$

$$\rightarrow \quad L(B^2 - H_o^2) = (2BH_L - B^2 - H_o^2)X^*$$

$$\therefore \quad \boxed{X^* = \frac{L(B^2 - H_o^2)}{(2BH_L - B^2 - H_o^2)}} \leftarrow$$

2. SOLUTION FOR DISCHARGE

DERIVE THE EXPRESSION FOR Q FROM (1) :

$$\frac{Q}{W} = \frac{K}{2}(B^2 - H_o^2)\frac{(2BH_L - B^2 - H_o^2)}{L(B^2 - H_o^2)}$$

$$\rightarrow \quad \boxed{\frac{Q}{W} = \frac{K}{2}\frac{(2BH_L - B^2 - H_o^2)}{L}}$$

CHECK : DERIVE THE EXPRESSION FOR Q FROM (2) :

$$\frac{Q}{W} = KB\frac{(H_L - B)}{\left(L - \left[\frac{L(B^2 - H_o^2)}{(2BH_L - B^2 - H_o^2)}\right]\right)}$$

$$= KB\frac{(H_L - B)}{\left(\frac{2BH_L \cdot L - LB^2 - \cancel{LH_o^2} - LB^2 + \cancel{LH_o^2}}{(2BH_L - B^2 - H_o^2)}\right)}$$

$$= KB\frac{(H_L - B)(2BH_L - B^2 - H_o^2)}{\left(2BH_L \cdot L - 2LB^2\right)}$$

$$= K\cancel{B}\frac{\cancel{(H_L - B)}(2BH_L - B^2 - H_o^2)}{2\cancel{B}L\,\cancel{(H_L - B)}}$$

$$= \frac{K}{2}\frac{(2BH_L - B^2 - H_o^2)}{L} \quad ✓$$

3. Solutions for head profiles

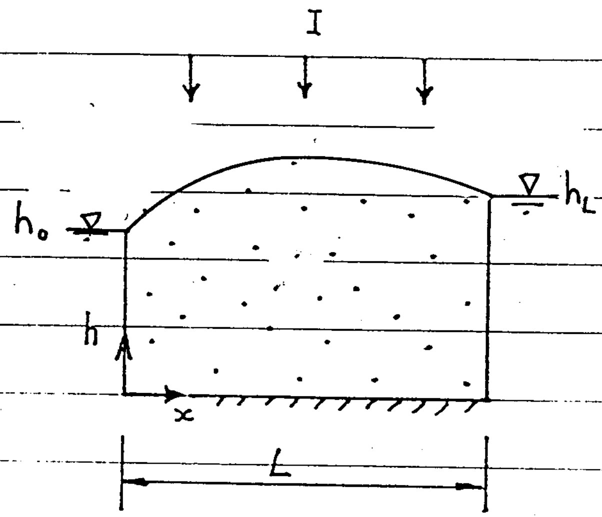
a) Unconfined: $0 \leq x \leq X^*$

$$\boxed{h^2 = H_o^2 + (B^2 - H_o^2)\frac{x}{X^*}}$$

b) Confined: $X^* \leq x \leq L$

$$\boxed{h = B + (H_L - B)\frac{(x - X^*)}{(L - X^*)}}$$

STEADY 1D UNCONFINED FLOW: DUPUIT-FORCHHEIMER SOLUTION



$0 \le x \le L$

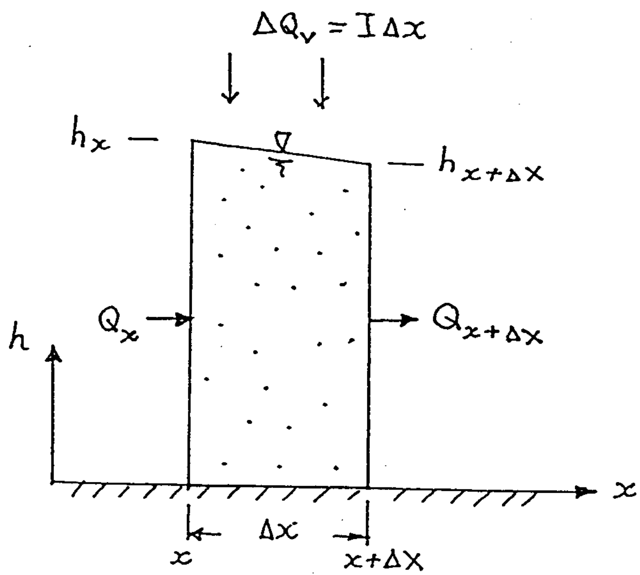
Key assumptions:

- Resistance to vertical flow is negligible (Dupuit assumption);
- Flat base;
- Uniform hydraulic conductivity, K; and
- Uniform recharge, I.

Derivation:

1. Governing equation
2. General solution
3. Particular case: Specified heads at $x = 0$ and $x = L$

1. GOVERNING EQUATION



Writing a flow balance for the slice of aquifer :

$$Q_{x+\Delta x} = Q_x + \Delta Q_v$$

or, $$Q_{x+\Delta x} - Q_x = \Delta Q_v$$

Now $$Q_x = h_x \, q_x$$

$$Q_{x+\Delta x} = h_{x+\Delta x} \cdot q_{x+\Delta x}$$

and $$\Delta Q_v = I \, \Delta x$$

Substituting into the flow balance :

$$h_{x+\Delta x} \cdot q_{x+\Delta x} - h_x q_x = I \Delta x$$

Noting that $\quad hq\big|_{x+\Delta x} = hq\big|_x + \frac{d}{dx}(hq)\big|_x \Delta x$

the flow balance becomes :

$$\left(h_x q_x + \frac{d}{dx}(hq)\Big|_x \Delta x \right) - h_x q_x = I \Delta x$$

Simplifying,

$$\frac{d}{dx}(hq) \Delta x = I \Delta x$$

Dividing through by Δx :

$$\frac{d}{dx}(hq) = I$$

The Darcy flux q is given by Darcy's Law :

$$q = -K \frac{dh}{dx}$$

Substituting for q in the statement of mass balance yields:

$$-\frac{d}{dx}\left(h\,K\,\frac{dh}{dx}\right) = I$$

Re-arranging:

$$\frac{d}{dx}\left(K\,h\,\frac{dh}{dx}\right) + I = 0$$

For uniform hydraulic conductivity:

$$\boxed{K\,\frac{d}{dx}\left(h\,\frac{dh}{dx}\right) + I = 0} \qquad (1)$$

2. <u>General solution</u>:

Let $\phi = h^2 \longrightarrow h = \phi^{1/2}$

$$\frac{dh}{dx} = \frac{1}{2}\phi^{-1/2}\frac{d\phi}{dx}$$

Substituting into the governing equation:

$$K\frac{d}{dx}\left(\phi^{1/2}\frac{1}{2}\phi^{-1/2}\frac{d\phi}{dx}\right) + I = 0$$

which reduces to:

$$K\frac{d}{dx}\left(\frac{d\phi}{dx}\right) + 2I = 0$$

or

$$K\frac{d}{dx}\left(\frac{d\phi}{dx}\right) = -2I$$

Integrating wrt x :

$$\frac{d\phi}{dx} = -\frac{2Ix}{K} + C_1$$

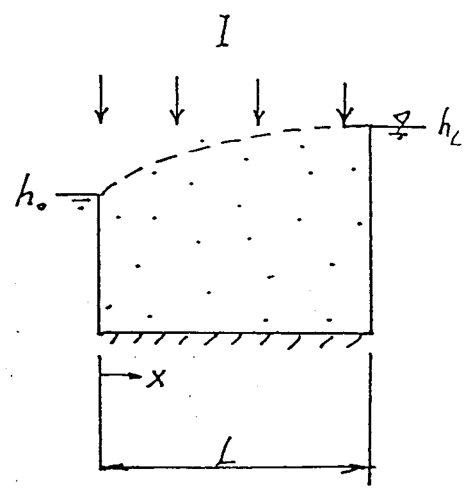
Integrating a second time wrt x :

$$\boxed{\phi = h^2 = -\frac{I}{K}x^2 + C_1 x + C_2} \qquad (2)$$

The coefficients C_1 and C_2 are evaluated by considering the boundary conditions for particular cases. In the following section we will consider such case

3. Particular case:

FIXED HEADS AT BOTH ENDS



$$h(0) = h_o$$
$$h(L) = h_L$$

$$\therefore \ \phi(0) = h_o^2 = C_2$$

and $\phi(L) = h_L^2 = -\frac{I}{K}L^2 + C_1 L + h_o^2$

$$\rightarrow \ C_1 = \frac{(h_L^2 - h_o^2)}{L} + \frac{IL}{K}$$

h_o may be smaller, the same, or larger than h_L.

Substituting for C_1 and C_2 in the general solution:

$$\phi = -\frac{I}{K}x^2 + \left[\frac{(h_L^2 - h_o^2)}{L} + \frac{I}{K}L\right]x + h_o^2$$

$$\therefore \ \phi = h_o^2 + \frac{(h_L^2 - h_o^2)}{L}x + \frac{I}{K}(L-x)x$$

$$\therefore \ \boxed{h = \left[h_o^2 + \frac{(h_L^2 - h_o^2)}{L}x + \frac{I}{K}(L-x)x\right]^{1/2}} \quad \text{——(3)}$$

- <u>Maximum head</u>

The maximum head occurs at x where $dh/dx = 0$.

$$\frac{dh}{dx} = \frac{d(h^2)}{dx}\frac{dh}{d(h^2)} = \frac{1}{2h}\left[\frac{(h_L^2-h_o^2)}{L} + \frac{IL}{K} - \frac{2Ix}{K}\right]$$

Now, setting the gradient to 0 yields the following condition for the location x^* of the maximum head:

$$0 = \frac{(h_L^2-h_o^2)}{L} + \frac{IL}{K} - \frac{2Ix^*}{K}$$

Solving for x^*:

$$x^* = \frac{K}{2I}\left[\frac{(h_L^2-h_o^2)}{L} + \frac{IL}{K}\right]$$

$$\therefore \boxed{x^* = \frac{K}{2I}\frac{(h_L^2-h_o^2)}{L} + \frac{L}{2}}$$

<u>CHECK</u>: If $h_L = h_o$ the problem is symmetric and the maximum head should occur at $x = L/2$.

$$x^*(h_L = h_o) = \frac{K}{2I}\frac{\cancelto{0}{(h_L^2-h_o^2)}}{L} + \frac{L}{2} = \frac{L}{2} \checkmark$$

Substituting for x^* in the solution for $h^2 (= \phi)$:

$$h_m^2 = h_o^2 + \frac{(h_L^2 - h_o^2)}{L}\left(\frac{K}{2I}\frac{(h_L^2-h_o^2)}{L} + \frac{L}{2}\right)$$

$$+ \frac{I}{K}\left(L - \left(\frac{K}{2I}\frac{(h_L^2-h_o^2)}{L} + \frac{L}{2}\right)\right)\left(\frac{K}{2I}\frac{(h_L^2-h_o^2)}{L} + \frac{L}{2}\right)$$

Simplifying :

$$h_m^2 = h_o^2 + \frac{(h_L^2 - h_o^2)}{L}\left(\frac{K}{2I}\frac{(h_L^2-h_o^2)}{L} + \frac{L}{2}\right)$$

$$+ \frac{I}{K}\left(\frac{L}{2} - \frac{K}{2I}\frac{(h_L^2-h_o^2)}{L}\right)\left(\frac{K}{2I}\frac{(h_L^2-h_o^2)}{L} + \frac{L}{2}\right)$$

Expanding :

$$h_m^2 = h_o^2 + \frac{K}{2I}\left(\frac{(h_L^2-h_o^2)}{L}\right)^2 + \frac{(h_L^2-h_o^2)}{2}$$

$$+ \frac{I}{K}\frac{L^2}{4} - \frac{I}{K}\left(\frac{K}{2I}\frac{(h_L^2-h_o^2)}{L}\right)^2$$

Simplifying :

$$\boxed{h_m^2 = h_o^2 + \frac{(h_L^2-h_o^2)}{2} + \frac{I}{K}\frac{L^2}{4} + \frac{K}{4I}\left(\frac{(h_L^2-h_o^2)}{L}\right)^2}$$

Generalization of conditions for maximum head

The maximum head can occur at $x=0$, $x=L$, or somewhere between 0 and L.

- If h_o is sufficiently large, the maximum head will occur at $x=0$.
- If h_L is sufficiently large, the maximum head will occur at $x=L$.

Derivation of conditions for the location of x^*, the location of h_{max}:

Divide x^* through by L:

$$\frac{x^*}{L} = \frac{K}{2I}\frac{(h_L^2-h_o^2)}{L^2} + \frac{1}{2}$$

a) If $\frac{K}{2I}\frac{(h_L^2-h_o^2)}{L^2} < -\frac{1}{2}$, $x^*<0$; h_{max} occurs at $x=0$
$h_{max} = h_o$

b) If $\frac{K}{2I}\frac{(h_L^2-h_o^2)}{L^2} > +\frac{1}{2}$, $x^* > L$; h_{max} occurs at $x=L$
$h_{max} = h_L$

c) If $\left|\frac{K}{2I}\frac{(h_L^2-h_o^2)}{L^2}\right| < \frac{1}{2}$, h_{max} occurs between $x=0$ and $x=L$
x^* given by (4)
h_{max} given by (5)

- Discharge rate

$$Q_x = -Kh\frac{dh}{dx}$$

$$= -\frac{K}{2}\frac{d(h^2)}{dx}$$

$$\therefore \quad Q_x = -\frac{K}{2}\left[\frac{(h_L^2-h_o^2)}{L} + \frac{IL}{K} - \frac{2Ix}{K}\right]$$

In particular,

$$@\ x=0: \quad Q_o = -\frac{K}{2}\left[\frac{(h_L^2-h_o^2)}{L} + \frac{IL}{K}\right]$$

$$@\ x=L: \quad Q_L = -\frac{K}{2}\left[\frac{(h_L^2-h_o^2)}{L} - \frac{IL}{K}\right]$$

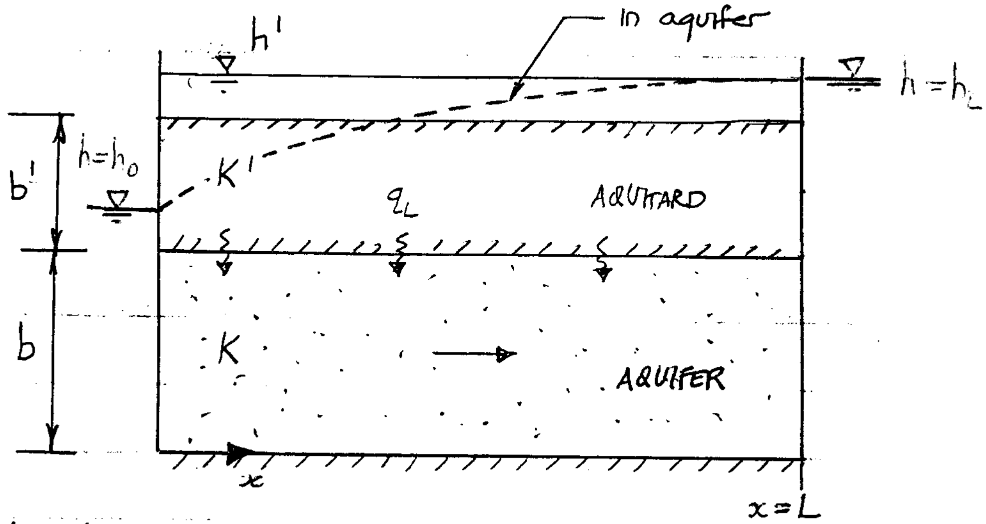
(6, 6

For $h_o = 0.0$, (6A) yields: $\quad Q_o = -\frac{K}{2}\left[\frac{h_L^2}{L} + \frac{IL}{K}\right]$

CJN 2008/09/16
MODEL 5 DERIVATION

STEADY 1D FLOW IN A LEAKY AQUIFER

1. CONCEPTUAL MODEL:



KEY ASSUMPTIONS:

- STEADY FLOW
- 1D HORIZONTAL FLOW IN AQUIFER
- 1D VERTICAL FLOW IN AQUITARD

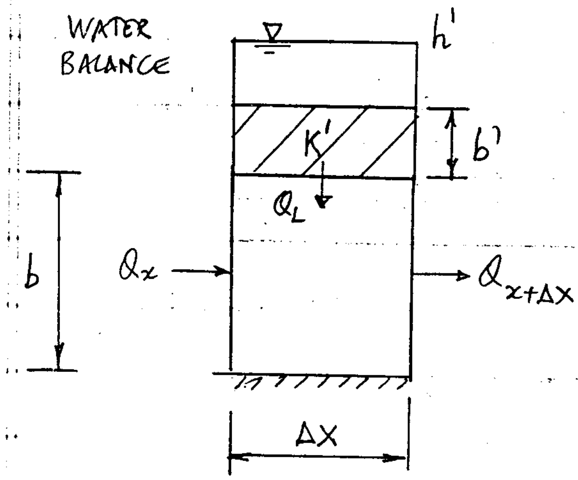
$\longrightarrow K' \ll K$

- UNIFORM AQUIFER TRANSMISSIVITY, Kb
- UNIFORM AQUITARD PROPERTIES, K' AND b'
- UNIFORM HEAD AT TOP OF AQUITARD, h'

LEAKAGE, $q_L = +K' \left(\frac{h' - h(x)}{b'} \right)$

CHECK SIGN:
IF $h > h'$, aquifer 'leaks' into aquitard
$\therefore$ sink, $q_L < 0$ ✓
IF $h < h'$, aquitard 'leaks' into aquifer
$\therefore$ source, $q_L > 0$ ✓

2. DERIVATION OF GOVERNING EQUATION



$$Q_{x+\Delta x} = Q_x + \Delta Q_L$$

$$q_{x+\Delta x} b = q_x b + \Delta Q_L$$

$$\Delta Q_L = -\frac{K'}{b'}(h-h')\Delta x$$

$$\therefore\ q_{x+\Delta x} b = q_x b + \frac{K'}{b'}(h'-h)\Delta x$$

Noting that $q_{x+\Delta x} = q_x + \frac{dq}{dx}\Delta x$

$$\therefore\ \left(q_x + \frac{dq}{dx}\Delta x\right) b = q_x b + \frac{K'}{b'}(h'-h)\Delta x$$

Simplifying:

$$\frac{dq}{dx}\Delta x b = \frac{K'}{b'}(h'-h)\Delta x$$

Dividing through Δx:

$$b\frac{dq}{dx} = \frac{K'}{b'}(h'-h)$$

Darcy's Law: $q = -K\frac{dh}{dx}$

$$\therefore -b\frac{d}{dx}\left(K\frac{dh}{dx}\right) - \frac{K'}{b'}(h'-h) = 0$$

or $$\boxed{Kb\frac{d^2h}{dx^2} + \frac{K'}{b'}(h'-h) = 0} \quad \text{—(1)}$$

3. GENERAL SOLUTION:

WRITING THE GOVERNING EQUATION IN STANDARD FORM:

$$\frac{d^2h}{dx^2} - \frac{(K'/b')}{Kb} h = -\frac{(K'/b')}{Kb} h'$$

THE GOVERNING EQUATION IS A LINEAR, SECOND-ORDER, NONHOMOGENEOUS ODE WITH CONSTANT COEFFICIENTS. THE GENERAL SOLUTION CAN BE WRITTEN AS:

$$h = h_H + h_p$$

WHERE h_H AND h_p ARE THE HOMOGENEOUS AND PARTICULAR SOLUTIONS.

THE HOMOGENEOUS SOLUTION IS:

$$h_H = A \,\mathrm{EXP}\{m^+ x\} + B \,\mathrm{EXP}\{m^- x\}$$

WHERE $\quad m^{\pm} = \pm \left[\dfrac{(K'/b')}{Kb} \right]^{1/2}$

LET US CALL $\quad \lambda = \left[\dfrac{Kb}{(K'/b')} \right]^{1/2} \qquad$ ASSUMING $(K'/b') \neq 0$

$$\therefore \quad h_H = A \,\mathrm{EXP}\left\{ \frac{x}{\lambda} \right\} + B \,\mathrm{EXP}\left\{ -\frac{x}{\lambda} \right\}$$

THE PARTICULAR SOLUTION CAN BE DERIVED USING THE SHORT-CUT METHOD OF OPERATORS :

$$h_p = \mathrm{EXP}\{-P_1 x\} \int^{x} \mathrm{EXP}\{P_1 \xi\} \left[\mathrm{EXP}\{-P_2 \xi\} \int^{\xi} \mathrm{EXP}\{P_2 \chi\}\, h(\chi)\, d\chi \right] d\xi$$

Here $\quad P_1 = \dfrac{1}{\lambda} \quad ; \quad P_2 = -\dfrac{1}{\lambda} \quad ; \quad h = -\dfrac{(K'/b)}{Kb} h' = \dfrac{-h'}{\lambda^2}$

$$h_p = \mathrm{EXP}\left\{-\frac{x}{\lambda}\right\} \int^{x} \mathrm{EXP}\left\{\frac{\xi}{\lambda}\right\} \left[\mathrm{EXP}\left\{\frac{\xi}{\lambda}\right\} \int^{\xi} \mathrm{EXP}\left\{-\frac{\chi}{\lambda}\right\} \left(-\frac{h'}{\lambda^2}\right) d\chi \right] d\xi$$

$$= -\frac{h'}{\lambda^2} \mathrm{EXP}\left\{-\frac{x}{\lambda}\right\} \int^{x} \mathrm{EXP}\left\{\frac{\xi}{\lambda}\right\} \left[\mathrm{EXP}\left\{\frac{\xi}{\lambda}\right\} \left(-\lambda\, \mathrm{EXP}\left\{-\frac{\xi}{\lambda}\right\} \right) \right] d\xi$$

$$= \frac{h'}{\lambda} \mathrm{EXP}\left\{-\frac{x}{\lambda}\right\} \int^{x} \mathrm{EXP}\left\{\frac{\xi}{\lambda}\right\} d\xi$$

$$= \frac{h'}{\lambda} \mathrm{EXP}\left\{-\frac{x}{\lambda}\right\} \left(\lambda\, \mathrm{EXP}\left\{\frac{x}{\lambda}\right\} \right)$$

$$= h'$$

THE GENERAL SOLUTION IS THEREFORE :

$$\boxed{h = A \exp\left\{\frac{x}{\lambda}\right\} + B \exp\left\{-\frac{x}{\lambda}\right\} + h'} \qquad \text{—(2)}$$

4. PARTICULAR SOLUTION

SPECIFIED HEADS IN AQUIFER AT $x=0$ AND $x=L$

$$h(0) = h_o \qquad \text{—(3a)}$$
$$h(L) = h_L \qquad \text{—(3b)}$$

EVALUATING THE BOUNDARY CONDITIONS WITH THE GENERAL SOLUTION:

$$h(0) = h_o = A \exp\left\{\frac{x}{\lambda}\right\} + B \exp\left\{-\frac{x}{\lambda}\right\} + h' \Bigg|_{x=0}$$

$$= A + B + h' \qquad \text{—(i)}$$

$$h(L) = h_L = A \exp\left\{\frac{x}{\lambda}\right\} + B \exp\left\{-\frac{x}{\lambda}\right\} + h' \Bigg|_{x=L}$$

$$= A \exp\left\{\frac{L}{\lambda}\right\} + B \exp\left\{-\frac{L}{\lambda}\right\} + h' \qquad \text{—(ii)}$$

SOLVING FOR A FROM (i):

$$A = h_o - h' - B$$

SUBSTITUTING FOR A IN (ii):

$$h_L = \left[h_o - h' - B\right] \exp\left\{\frac{L}{\lambda}\right\} + B \exp\left\{-\frac{L}{\lambda}\right\} + h'$$

COLLECTING TERMS :

$$h_L - h' - [h_o - h'] \, \mathrm{EXP}\left\{\frac{L}{\lambda}\right\} = -B \, \mathrm{EXP}\left\{\frac{L}{\lambda}\right\} + B \, \mathrm{EXP}\left\{-\frac{L}{\lambda}\right\}$$

SOLVING FOR B :

$$B = \frac{h_L - h' - [h_o - h'] \, \mathrm{EXP}\left\{\frac{L}{\lambda}\right\}}{\mathrm{EXP}\left\{-\frac{L}{\lambda}\right\} - \mathrm{EXP}\left\{\frac{L}{\lambda}\right\}}$$

DIVIDING THROUGH BY $\mathrm{EXP}\left\{\frac{L}{\lambda}\right\}$:

$$B = \frac{[h_L - h'] \, \mathrm{EXP}\left\{-\frac{L}{\lambda}\right\} - [h_o - h']}{\mathrm{EXP}\left\{-\frac{2L}{\lambda}\right\} - 1}$$

$$\therefore \quad A = h_o - h' - \left[\frac{[h_L - h'] \, \mathrm{EXP}\left\{-\frac{L}{\lambda}\right\} - [h_o - h']}{\mathrm{EXP}\left\{-\frac{2L}{\lambda}\right\} - 1}\right]$$

$$= \frac{[h_o - h']\left[\mathrm{EXP}\left\{-\frac{2L}{\lambda}\right\} - 1\right] - \left[[h_L - h'] \, \mathrm{EXP}\left\{-\frac{L}{\lambda}\right\} - [h_o - h']\right]}{\mathrm{EXP}\left\{-\frac{2L}{\lambda}\right\} - 1}$$

SIMPLIFYING :

$$A = \frac{[h_o - h'] \, \mathrm{EXP}\left\{-\frac{2L}{\lambda}\right\} - [h_L - h'] \, \mathrm{EXP}\left\{-\frac{L}{\lambda}\right\}}{\mathrm{EXP}\left\{-\frac{2L}{\lambda}\right\} - 1}$$

THE FINAL SOLUTION IS ~~THEREFORE~~ :

$$\boxed{\begin{aligned} h = & \left(\frac{[h_o - h'] \, \mathrm{EXP}\left\{-\frac{2L}{\lambda}\right\} - [h_L - h'] \, \mathrm{EXP}\left\{-\frac{L}{\lambda}\right\}}{\mathrm{EXP}\left\{-\frac{2L}{\lambda}\right\} - 1}\right) \mathrm{EXP}\left\{\frac{x}{\lambda}\right\} \\ & + \left(\frac{[h_L - h'] \, \mathrm{EXP}\left\{-\frac{L}{\lambda}\right\} - [h_o - h']}{\mathrm{EXP}\left\{-\frac{2L}{\lambda}\right\} - 1}\right) \mathrm{EXP}\left\{-\frac{x}{\lambda}\right\} \\ & + h' \end{aligned}} \quad (4)$$

5. <u>CHECK:</u> Does the solution satisfy the boundary conditions?

1. @ $x = 0$:

$$h = \left(\frac{[h_o - h'] \, \mathrm{EXP}\left\{-\frac{2L}{\lambda}\right\} - [h_L - h'] \, \mathrm{EXP}\left\{-\frac{L}{\lambda}\right\}}{\mathrm{EXP}\left\{-\frac{2L}{\lambda}\right\} - 1} \right)$$

$$+ \left(\frac{[h_L - h'] \, \mathrm{EXP}\left\{-\frac{L}{\lambda}\right\} - [h_o - h']}{\mathrm{EXP}\left\{-\frac{2L}{\lambda}\right\} - 1} \right)$$

$$+ h'$$

$$= \Bigg([h_o - h'] \, \mathrm{EXP}\left\{-\frac{2L}{\lambda}\right\} - [h_L - h'] \, \mathrm{EXP}\left\{-\frac{L}{\lambda}\right\}$$

$$+ [h_L - h'] \, \mathrm{EXP}\left\{-\frac{L}{\lambda}\right\} - [h_o - h'] \Bigg) \cdot \frac{1}{\mathrm{EXP}\left\{-\frac{2L}{\lambda}\right\} - 1}$$

$$+ h'$$

$$= \Bigg([h_o - h'] \, \mathrm{EXP}\left\{-\frac{2L}{\lambda}\right\} + \overbrace{[-h_L + h' + h_L - h']}^{0} \, \mathrm{EXP}\left\{-\frac{L}{\lambda}\right\}$$

$$- [h_o - h'] \Bigg) \cdot \frac{1}{\mathrm{EXP}\left\{-\frac{2L}{\lambda}\right\} - 1} + h'$$

$$= \frac{[h_o - h'] \left(\mathrm{EXP}\left\{-\frac{2L}{\lambda}\right\} - 1 \right)}{\mathrm{EXP}\left\{-\frac{2L}{\lambda}\right\} - 1} + h'$$

$$= h_o \quad \checkmark$$

2. @ $x=L$:

$$h = \left(\frac{[h_0-h'] \, EXP\{-\frac{2L}{\lambda}\} - [h_L-h'] \, EXP\{-\frac{L}{\lambda}\}}{EXP\{-\frac{2L}{\lambda}\} - 1} \right) EXP\{\frac{L}{\lambda}\}$$

$$+ \left(\frac{[h_L-h'] \, EXP\{-\frac{L}{\lambda}\} - [h_0-h']}{EXP\{-\frac{2L}{\lambda}\} - 1} \right) EXP\{-\frac{L}{\lambda}\}$$

$$+ h'$$

$$= \left(\overset{\checkmark}{[h_0-h'] \, EXP\{-\frac{L}{\lambda}\}} - [h_L-h'] + [h_L-h'] \, EXP\{-\frac{2L}{\lambda}\} - \overset{\checkmark}{[h_0-h'] \, EXP\{-\frac{L}{\lambda}\}} \right) \cdot \frac{1}{EXP\{-\frac{2L}{\lambda}\} - 1}$$

$$+ h'$$

$$= -[h_L-h'] \left(1 - EXP\{-\frac{2L}{\lambda}\} \right) \cdot \frac{1}{EXP\{-\frac{2L}{\lambda}\} - 1}$$

$$+ h'$$

$$= [h_L-h'] + h'$$

$$= h_L \quad \checkmark$$

6. MORE ROBUST FORMULATION

$$\boxed{h = \left(\frac{[h_o - h'] \, \mathrm{EXP}\left\{-\frac{(2L-x)}{\lambda}\right\} - [h_L - h'] \, \mathrm{EXP}\left\{-\frac{(L-x)}{\lambda}\right\}}{\mathrm{EXP}\left\{-\frac{2L}{\lambda}\right\} - 1}\right) + \left(\frac{[h_L - h'] \, \mathrm{EXP}\left\{-\frac{(L+x)}{\lambda}\right\} - [h_o - h'] \, \mathrm{EXP}\left\{-\frac{x}{\lambda}\right\}}{\mathrm{EXP}\left\{-\frac{2L}{\lambda}\right\} - 1}\right) + h'} \quad (5)$$

7. CHECK: Does the more robust form of the solution satisfy the boundary conditions?

1. @ $x = 0$:

$$h = \left(\frac{[h_o - h'] \, \mathrm{EXP}\left\{-\frac{2L}{\lambda}\right\} - [h_L - h'] \, \mathrm{EXP}\left\{-\frac{L}{\lambda}\right\}}{\mathrm{EXP}\left\{-\frac{2L}{\lambda}\right\} - 1}\right) + \left(\frac{[h_L - h'] \, \mathrm{EXP}\left\{-\frac{L}{\lambda}\right\} - [h_o - h']}{\mathrm{EXP}\left\{-\frac{2L}{\lambda}\right\} - 1}\right) + h'$$

$$= \left(\frac{h_o\left[\mathrm{EXP}\left\{-\frac{2L}{\lambda}\right\} - 1\right] + h_L\left[-\mathrm{EXP}\left\{-\frac{L}{\lambda}\right\} + \mathrm{EXP}\left\{-\frac{L}{\lambda}\right\}\right]^{\nearrow 0} + h'\left[-\mathrm{EXP}\left\{-\frac{2L}{\lambda}\right\} + \mathrm{EXP}\left\{-\frac{L}{\lambda}\right\} - \mathrm{EXP}\left\{-\frac{L}{\lambda}\right\} + 1\right]}{\mathrm{EXP}\left\{-\frac{2L}{\lambda}\right\} - 1}\right) + h'$$

$$= \frac{\left(h_0\left[EXP\left\{-\frac{2L}{\lambda}\right\}-1\right]-h'\left[EXP\left\{-\frac{2L}{\lambda}\right\}-1\right]\right)}{EXP\left\{-\frac{2L}{\lambda}\right\}-1} + h'$$

$$= (h_0 - h') + h' = h_0 \quad \checkmark$$

2. @ $x = L$: $h = \left(\frac{[h_0-h']EXP\{-\frac{L}{\lambda}\} - [h_L-h']\cancel{EXP\{\theta\}}}{EXP\{-\frac{2L}{\lambda}\}-1}\right)$

$$+ \left(\frac{[h_L-h']EXP\{-\frac{2L}{\lambda}\} - [h_0-h']EXP\{-\frac{x}{\lambda}\}}{EXP\{-\frac{2L}{\lambda}-1\}}\right)$$

$$+ h'$$

$\cancel{-[h_0-h']EXP\{-\frac{L}{\lambda}\}}$

$$= \frac{\cancel{[h_0-h']EXP\{-\frac{L}{\lambda}\}} - [h_L-h'] + [h_L-h']EXP\{-\frac{2L}{\lambda}\}}{EXP\{-\frac{2L}{\lambda}\}-1}$$

$$+ h'$$

$$= \frac{-[h_L-h']\left[1-EXP\left\{-\frac{2L}{\lambda}\right\}\right]}{EXP\left\{-\frac{2L}{\lambda}\right\}-1} + h'$$

$$= +[h_L-h'] + h'$$

$$= h_L \quad \checkmark$$

8. Discharge at x=0

$$Q(0) = -K \cdot b \frac{dh}{dx}(0)$$

$$Q(0) = -K \cdot b \left(\frac{\dfrac{(h_0 - \overset{h'}{h_L})EXP\left\{-\dfrac{2L}{\lambda}\right\}}{\lambda} - \dfrac{(h_L - h')EXP\left\{-\dfrac{L}{\lambda}\right\}}{\lambda}}{EXP\left\{-\dfrac{2L}{\lambda}\right\} - 1} + \frac{-\dfrac{(h_L - h')EXP\left\{-\dfrac{L}{\lambda}\right\}}{\lambda} + \dfrac{(h_0 - h')}{\lambda}}{EXP\left\{-\dfrac{2L}{\lambda}\right\} - 1} \right)$$

$$\therefore \boxed{Q = -Kb \cdot \frac{1}{\lambda} \cdot \frac{1}{EXP\left\{-\frac{2L}{\lambda}\right\} - 1} \left[(h_0 - h')\left(1 + EXP\left\{-\frac{2L}{\lambda}\right\}\right) - 2(h_L - h')EXP\left\{-\frac{L}{\lambda}\right\} \right]}$$

SPECIAL CASE : $h_L = h'$

For the special case of $h_L = h'$, the solution for the discharge reduces to:

$$Q = -Kb \cdot \frac{1}{\lambda} \frac{1}{EXP\left\{-\frac{2L}{\lambda}\right\} - 1} \left[(h_0 - h_L)\left(1 + EXP\left\{-\frac{2L}{\lambda}\right\}\right) \right]$$

Simplifying:

$$Q = +\frac{Kb}{\lambda} (h_0 - h_L) \frac{\left(1 + EXP\left\{-\frac{2L}{\lambda}\right\}\right)}{\left(1 - EXP\left\{-\frac{2L}{\lambda}\right\}\right)}$$

Re-arranging slightly:

$$\boxed{Q = -\frac{Kb}{\lambda} (h_L - h_0) \frac{\left(1 + EXP\left\{-\frac{2L}{\lambda}\right\}\right)}{\left(1 - EXP\left\{-\frac{2L}{\lambda}\right\}\right)}}$$

APPENDIX B

MODEL 6

DERIVATION

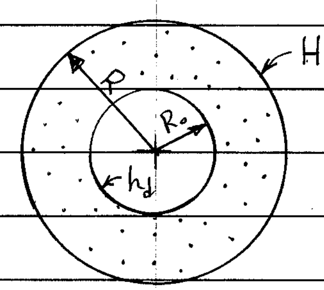
ANALYTICAL SOLUTION FOR RADIAL CONFINED FLOW TO A CIRCULAR EXCAVATION

1. SOLUTION FOR HEAD

$$\frac{1}{r}\frac{d}{dr}\left(KD\, r\frac{dh}{dr}\right) = 0 \qquad ; \quad R_0 \le r \le R$$

BOUNDARY CONDITIONS:

$$h(R_0) = h_d$$
$$h(R) = H$$



For homogeneous K and D we can divide through by $\frac{KD}{r}$ to obtain:

$$\frac{d}{dr}\left(r\frac{dh}{dr}\right) = 0$$

Integrating once wrt r:

$$r\frac{dh}{dr} = C_1$$

Integrating a second time wrt r:

$$h = C_1 \ln r + C_2$$

The coefficients are determined by considering the boundary conditions:

i) $\underline{r = R_0}$

$$h_d = C_1 \ln\{R_0\} + C_2$$

ii) $\underline{r = R}$

$$H = C_1 \ln\{R\} + C_2$$

$$\rightarrow \quad C_1 = \frac{H - h_d}{\ln\left\{\frac{R}{R_0}\right\}}$$

$$C_2 = H - \left[\frac{H - h_d}{\ln\left\{\frac{R}{R_0}\right\}}\right]\ln\{R\}$$

Substituting for C_1 and C_2 :

$$h = \left[\frac{H - h_d}{\ln\left\{\frac{R}{R_o}\right\}}\right] \ln\{r\} + \left\{ H - \left[\frac{H - h_d}{\ln\left\{\frac{R}{R_o}\right\}}\right] \ln\{R\} \right\}$$

Collecting terms :

$$h = H - \left[\frac{H - h_d}{\ln\left\{\frac{R}{R_o}\right\}}\right] \ln\left\{\frac{R}{r}\right\}$$

Simplifying :

$$\boxed{h = H - (H - h_d)\frac{\ln\left\{\frac{R}{r}\right\}}{\ln\left\{\frac{R}{R_o}\right\}}}$$

CHECK :

i) At $r = R_o$: $h = H - (H - h_d)$

$= h_d$ ✓

ii) At $r = R$: $h = H$ ✓

2. SOLUTION FOR DISCHARGE

$$Q = -2\pi R_o K D \left.\frac{dh}{dr}\right|_{r=R_o}$$

$$= +2\pi R_o K D \frac{(H-h_d)}{\ln\left\{\frac{R}{R_o}\right\}} \left(\frac{r}{R}\right)\left(-\frac{R}{r^2}\right)\Bigg|_{r=R_o}$$

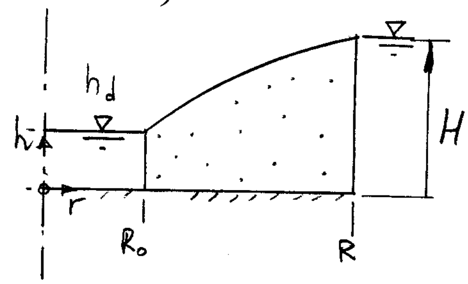
$$\therefore \boxed{Q = -2\pi K D \frac{(H-h_d)}{\ln\left\{\frac{R}{R_o}\right\}}}$$

Negative sign denotes flow into the excavation when $h_d < H$.

ANALYTICAL SOLUTION FOR STEADY UNCONFINED RADIAL FLOW TO A CIRCULAR EXCAVATION

1. The governing equation for unconfined radial flow is:

$$\frac{1}{r}\frac{d}{dr}\left(K h r \frac{dh}{dr}\right) = 0 \quad ; \quad R_o \leq r \leq R$$



SUBJECT TO:

$$h(R_o) = h_d$$
$$h(R) = H$$

For homogeneous K we can divide through by $\frac{K}{r}$ to obtain:

$$\frac{d}{dr}\left(h r \frac{dh}{dr}\right) = 0$$

Let us define a new variable $u = h^2$

$$\rightarrow \frac{du}{dr} = 2h\frac{dh}{dr}$$

The governing equation becomes:

$$\frac{d}{dr}\left(\frac{1}{2} r \frac{du}{dr}\right) = 0 \longrightarrow \frac{d}{dr}\left(r \frac{du}{dr}\right) = 0$$

Subject to:

$$u(R_o) = h_d^2$$
$$u(R) = H^2$$

The solution for u can be derived using the same procedure as was used to solve for head under confined conditions.

$$u = H^2 - (H^2 - h_d^2)\frac{\ln\{\frac{R}{r}\}}{\ln\{\frac{R}{R_o}\}}$$

i.e.,

$$\boxed{h = \left[H^2 - (H^2 - h_d^2)\frac{\ln\{\frac{R}{r}\}}{\ln\{\frac{R}{R_o}\}} \right]^{1/2}}$$

2. <u>SOLUTION FOR DISCHARGE</u>

$$Q = -2\pi R_o K\, h \frac{dh}{dr}\bigg|_{r=R_o}$$

$$= -2\pi R_o K \frac{1}{2}\frac{du}{dr}\bigg|_{r=R_o}$$

$$= +2\pi R_o K \frac{1}{2}\frac{(H^2 - h_d^2)}{\ln\{\frac{R}{R_o}\}}\left(\frac{r}{R}\right)\left(\frac{-R}{r^2}\right)\bigg|_{r=R_o}$$

$$\therefore \boxed{Q = -\pi K \frac{(H^2 - h_d^2)}{\ln\{\frac{R}{R_o}\}}}$$

CHECK:

As a simple check, we can expand the solution as:

$$Q = -2\pi K \frac{\frac{1}{2}(H-h_d)(H+h_d)}{\ln\left\{\frac{R}{R_o}\right\}}$$

Designating $\frac{1}{2}(H+h_d)$ as the average saturated thickness $\bar{D}$, we see that the unconfined solution can be written as:

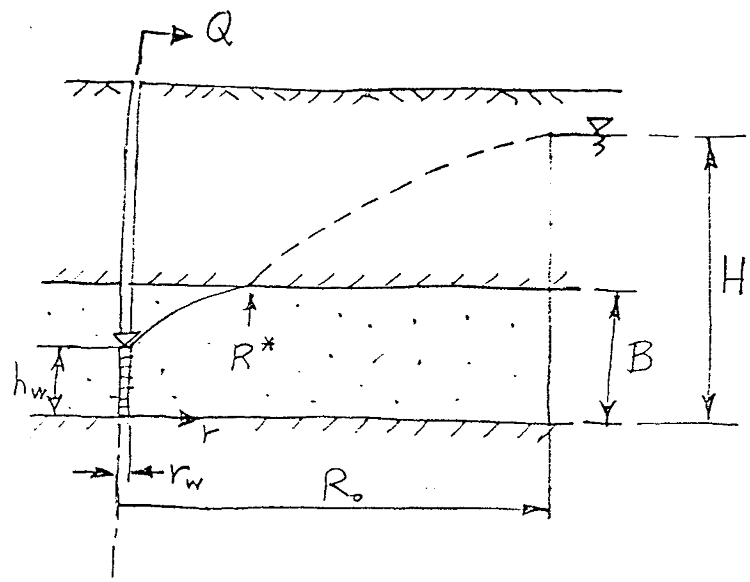
$$Q = -2\pi K \bar{D} \frac{(H-h_d)}{\ln\left\{\frac{R}{R_o}\right\}}$$

→ This is identical in form to the solution for confined conditions.

CJN 2011/11

Model 8:

ANALYSIS OF STEADY RADIAL FLOW TO A WELL WITH CONVERSION FROM CONFINED TO UNCONFINED CONDITIONS



DERIVATION:

i) $r_w \leq r \leq R^*$: Unconfined flow

$$Q = \pi K \frac{(B^2 - h_w^2)}{\ln\left\{\frac{R^*}{r_w}\right\}} \qquad -(1)$$

ii) $R^* \leq r \leq R_o$: Confined flow

$$Q = 2\pi K B \frac{(H - B)}{\ln\left\{\frac{R_o}{R^*}\right\}} \qquad -(2)$$

iii) Solve for R^*

Equating (1) and (2) :

$$Q = \pi K \frac{(B^2 - h_w^2)}{\ln\left\{\frac{R^*}{r_w}\right\}} = 2\pi K B \frac{(H-B)}{\ln\left\{\frac{R_o}{R^*}\right\}}$$

Simplifying :

$$\frac{(B^2 - h_w^2)}{\ln\left\{\frac{R^*}{r_w}\right\}} = 2B \frac{(H-B)}{\ln\left\{\frac{R_o}{R^*}\right\}}$$

Rearranging :

$$(B^2 - h_w^2) \ln\left\{\frac{R_o}{R^*}\right\} = 2B(H-B) \ln\left\{\frac{R^*}{r_w}\right\}$$

Expanding :

$$\therefore (B^2 - h_w^2)\left[\ln\{R_o\} - \ln\{R^*\}\right] = 2B(H-B)\left[\ln\{R^*\} - \ln\{r_w\}\right]$$

Collecting terms in $\ln\{R^*\}$:

$$\ln\{R^*\}\left[(B^2 - h_w^2) + 2B(H-B)\right] = (B^2 - h_w^2)\ln\{R_o\} + 2B(H-B)\ln\{r_w\}$$

Solving for $\ln\{R^*\}$:

$$\ln\{R^*\} = \frac{(B^2-h_w^2)\ln\{R_o\} + 2B(H-B)\ln\{r_w\}}{(B^2-h_w^2) + 2B(H-B)}$$

iv) Substituting for $\ln\{R^*\}$ in (1):

$$Q = \pi K \frac{(B^2-h_w^2)}{\left[\dfrac{(B^2-h_w^2)\ln\{R_o\} + 2B(H-B)\ln\{r_w\}}{(B^2-h_w^2) + 2B(H-B)}\right] - \ln\{r_w\}}$$

Expanding:

$$Q = \pi K \frac{(B^2-h_w^2)\left\{(B^2-h_w^2) + 2B(H-B)\right\}}{(B^2-h_w^2)\ln\{R_o\} + 2B(H-B)\ln\{r_w\} - \ln\{r_w\}\left\{(B^2-h_w^2) + 2B(H-B)\right\}}$$

Simplifying:

$$Q = \pi K \frac{(B^2-h_w^2)\left\{(B^2-h_w^2) + 2B(H-B)\right\}}{(B^2-h_w^2)\ln\{R_o\} - (B^2-h_w^2)\ln\{r_w\}}$$

$$\therefore \boxed{Q = \pi K \frac{\left\{(B^2-h_w^2) + 2B(H-B)\right\}}{\ln\left\{\frac{R_o}{r_w}\right\}}} \qquad (3)$$

CHECK :

Do we obtain the same result if we instead substitute for $\ln\{R^*\}$ in EQ^n (2) ?

$$Q = 2\pi KB \frac{(H-B)}{\ln\{R_o\} - \left[\dfrac{(B^2-h_w^2)\ln\{R_o\} + 2B(H-B)\ln\{r_w\}}{(B^2-h_w^2) + 2B(H-B)}\right]}$$

Expanding :

$$Q = 2\pi KB \frac{(H-B)\left\{(B^2-h_w^2) + 2B(H-B)\right\}}{\ln\{R_o\}\left\{(B^2-h_w^2) + 2B(H-B)\right\} - \left\{(B^2-h_w^2)\ln\{R_o\} + 2B(H-B)\ln\{r_w\}\right\}}$$

Simplifying :

$$Q = 2\pi KB \frac{(H-B)\left\{(B^2-h_w^2) + 2B(H-B)\right\}}{2B(H-B)\ln\{R_o\} - 2B(H-B)\ln\{r_w\}}$$

$$= 2\pi KB \frac{(H-B)\left\{(B^2-h_w^2) + 2B(H-B)\right\}}{2B(H-B)\ln\left\{\dfrac{R_o}{r_w}\right\}}$$

$$\therefore \quad Q = \pi K \frac{\left\{(B^2-h_w^2) + 2B(H-B)\right\}}{\ln\left\{\dfrac{R_o}{r_w}\right\}} \qquad \text{—(4)}$$

→ This is identical to EQ^n (3). ✓

Recalling the solution as presented

$$Q = \frac{\pi K \left(2BH - B^2 - h_w^2\right)}{\ln\left\{\frac{R_o}{r_w}\right\}}$$

Is the solution we have derived the same?

i.e., $$B^2 - h_w^2 + 2B\,(H-B) \stackrel{?}{=} 2BH - B^2 - h_w^2$$

Expanding the LHS:

$$B^2 - h_w^2 + 2BH - 2B^2 = 2BH - B^2 - h_w^2$$

This is identical to the RHS.

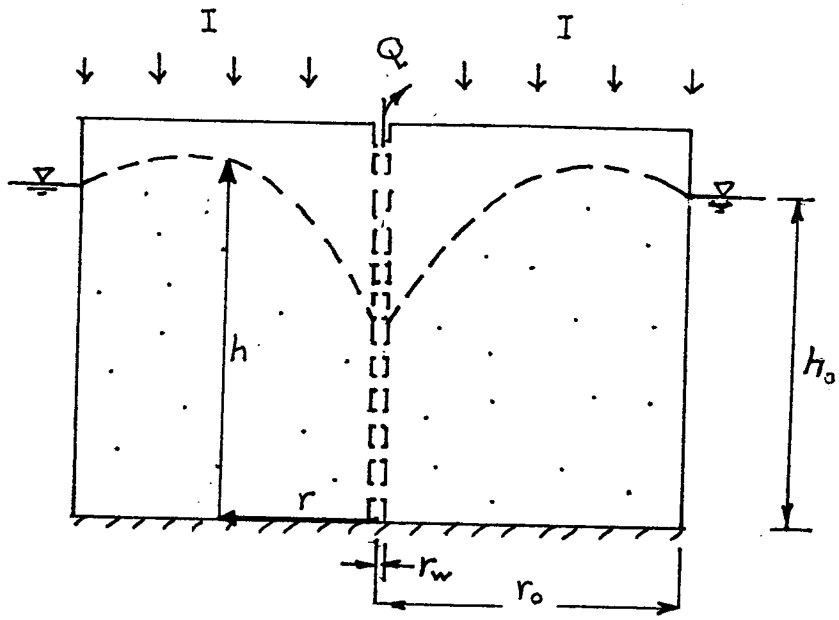
→ Yes, the solution we have derived <u>is</u> the same.

Q.E.D.

MODEL 9 :

SOLUTION FOR FLOW TO A SINGLE WELL WITH UNIFORM RECHARGE IN AN UNCONFINED AQUIFER : DUPUIT-FORCHHEIMER SOLUTION

Consider steady radial flow to a well in a Dupuit aquifer with a horizontal base, with uniform recharge across the top:

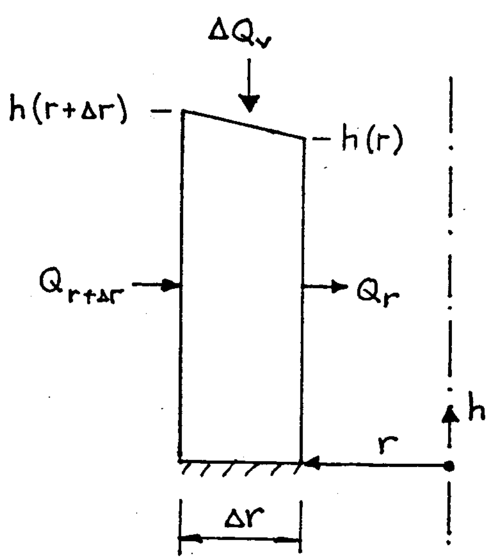


- well pumped at constant rate
- head at r_o remains consta at h_o

I. DERIVATION OF GOVERNING EQUATION (Dupuit-Forchheimer Model)

Define : Q = discharge rate, LT^{-3}
h = head above datum, L
r = radial distance, L
K = hydraulic conductivity, LT^{-1}
I = infiltration rate, LT^{-1}

Writing a flow balance for a slice of the aquifer :



$$Q_r = Q_{r+\Delta r} + \Delta Q_v$$

or, $$-\left[Q_{r+\Delta r} - Q_r \right] = \Delta Q_v$$

Now ;
$$Q_r = -2\pi r h q \Big|_r$$

$$Q_{r+\Delta r} = -2\pi (r+\Delta r) h q \Big|_{r+\Delta r}$$

where $q =$ Darcy flux $= -K \dfrac{\partial h}{\partial r}$

and $\Delta Q_V = I \cdot \pi \left[(r+\Delta r)^2 - r^2 \right]$

Substituting into the flow balance :

$$-\left[-2\pi (r+\Delta r) h q \Big|_{r+\Delta r} + 2\pi r h q \Big|_r \right] = I \pi \left[(r+\Delta r)^2 - r^2 \right]$$

Noting that :

$$hq \Big|_{r+\Delta r} = hq \Big|_r + \frac{d}{dr}(hq) \Big|_r \Delta r$$

the flow balance becomes :

$$\left[2\pi (r+\Delta r) \left(hq \Big|_r + \frac{d}{dr}(hq)\Big|_r \Delta r \right) - 2\pi r h q \Big|_r \right] = I \pi \left[(r+\Delta r)^2 - r^2 \right]$$

Expanding :

$$2\pi \left[\, r\,hq\big|_r + r\frac{d}{dr}(hq)\Big|_r \Delta r + \Delta r\; hq\big|_r + (\Delta r)^2 \frac{d}{dr}(hq)\Big|_r - r\,h\,q\big|_r \right] = I\pi \left[r^2 + 2r\Delta r + (\Delta r)^2 - r^2 \right]$$

Simplifying :

$$2 \left[r\frac{d}{dr}(hq)\Big|_r \Delta r + \Delta r\; hq\big|_r + (\Delta r)^2 \frac{d}{dr}(hq)\Big|_r \right]$$

$$= I \left[2r\Delta r + (\Delta r)^2 \right]$$

Dropping higher order terms and dividing through by $-2r\Delta r$:

$$\frac{d}{dr}(hq) + \frac{1}{r}(hq) = I$$

substituting for q yields the final form of the governing equation :

$$\boxed{\frac{d}{dr}\left(Kh\frac{dh}{dr}\right) + \frac{1}{r}Kh\frac{dh}{dr} + I = 0}$$

The following form of the governing equation is identical:

$$\boxed{\frac{1}{r}\frac{d}{dr}\left(rKh\frac{dh}{dr}\right) + I = 0}$$

Also :

let $u = h^2 \rightarrow h = u^{-1/2}$

$$\frac{dh}{dr} = \frac{1}{2h}\frac{d(h^2)}{dr} = \frac{1}{2u^{1/2}}\frac{du}{dr}$$

the governing eq^{n} becomes :

$$\frac{1}{r}\frac{d}{dr}\left(rKu^{1/2}\frac{1}{2u^{1/2}}\frac{du}{dr}\right) + I = 0$$

$$\rightarrow \quad \frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) + \frac{2I}{K} = 0$$

II. BOUNDARY CONDITIONS #1 : DISCHARGE CONTROL

i) $\underline{r = r_w}$:
$$\lim_{r \to r_w} -2\pi r K h \frac{dh}{dr} = -Q$$

INSIDE B.C.
→ DISCHARGE CONTROL

(Positive pumping rate yields flow inwards to the well)

$$Q = 2\pi r K h \frac{dh}{dr} \cong 2\pi \bar{r} K \bar{h} \left(\frac{h_{r+\Delta r} - h_r}{\Delta r} \right)$$

$$\therefore \quad h_{r+\Delta r} = \frac{Q}{2\pi \bar{r} K \bar{h}} \Delta r + h_r$$

For $Q > 0 \rightarrow h_{r+\Delta r} > h_r$, i.e., flow inwards

→ SIGN CONVENTION : $Q > 0$ for WITHDRAWAL

ii) $\underline{r = r_o}$: $h(r_o) = h_o$

OUTSIDE B.C.

III. SOLUTION

1. Assuming constant K, the governing equation can be re-written as:

$$\frac{d}{dr}\left(h\frac{dh}{dr}\right) + \frac{1}{r}h\frac{dh}{dr} = -\frac{I}{K}$$

Or, equivalently:

$$\frac{1}{r}\frac{d}{dr}\left(rh\frac{dh}{dr}\right) = -\frac{I}{K}$$

2. Defining $u = h^2 \longrightarrow h = u^{1/2}$

$$dh = \frac{1}{2}u^{-1/2}\,du$$

the governing equation becomes:

$$\frac{1}{r}\frac{d}{dr}\left(r\,u^{1/2}\,\frac{1}{2}u^{-1/2}\frac{du}{dr}\right) = -\frac{I}{K}$$

Simplifying:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = -2\frac{I}{K}$$

or,
$$\frac{d}{dr}\left(r\frac{du}{dr}\right) = -2\frac{I}{K}r$$

3. Integrating wrt r :

$$r \frac{du}{dr} = -2 \frac{I}{K} \frac{r^2}{2} + C_1$$

$$\therefore \quad \frac{du}{dr} = - \frac{I}{K} r + \frac{C_1}{r}$$

Integrating a second time wrt r :

$$u = - \frac{I}{K} \frac{r^2}{2} + C_1 \ln r + C_2$$ ← General solution

<u>check</u> : $\frac{du}{dr} = - \frac{I}{K} r + \frac{C_1}{r}$

$$\therefore \quad \frac{1}{r} \frac{d}{dr} \left(r \frac{du}{dr} \right) = \frac{1}{r} \frac{d}{dr} \left(- \frac{I}{K} r^2 + C_1 \right)$$

$$= \frac{1}{r} \left[- \frac{2Ir}{K} \right] = - \frac{2I}{K}$$ ✓

4. Determine the coefficients by evaluating the boundary conditions

i) $\underline{r = r_w}$:

Recalling that $h \frac{dh}{dr} = \frac{1}{2} \frac{du}{dr}$

the inner boundary condition can be written as :

$$\lim_{r \to r_w} 2 \pi r K \left(\frac{1}{2} \frac{du}{dr} \right) = Q$$

$$\therefore \lim_{r \to r_w} r \frac{du}{dr} = \frac{Q}{\pi K}$$

$$\hookrightarrow \lim_{r \to r_w} r \left(- \frac{I}{K} r + \frac{C_1}{r} \right) = \frac{Q}{\pi K}$$

$$\therefore C_1 = \frac{Q}{\pi K} + \frac{I r_w^2}{K}$$

ii) $\underline{r = r_o}$:

$$u(r_o) = h_o^2 = - \frac{I}{K} \frac{r_o^2}{2} + \left(\frac{Q}{\pi K} + \frac{I r_w^2}{K} \right) \ln r_o + C_2$$

$$\therefore \quad C_2 = h_o^2 + \frac{I}{K}\frac{r_o^2}{2} - \frac{Q}{\pi K}\ln r_o - \frac{I r_w^2}{K}\ln r_o$$

Substituting for C_1 and C_2 in the general solution yields :

$$u = -\frac{I}{K}\frac{r^2}{2} + \left(\frac{Q}{\pi K} + \frac{I r_w^2}{K}\right)\ln r$$

$$+ \left(h_o^2 + \frac{I}{K}\frac{r_o^2}{2} - \frac{Q}{\pi K}\ln r_o - \frac{I r_w^2}{K}\ln r_o\right)$$

Collecting terms and simplifying yields :

$$\boxed{h^2 = h_o^2 + \frac{I}{2K}\left(r_o^2 - r^2\right) - \frac{Q}{\pi K}\ln\left(\frac{r_o}{r}\right) - \frac{I r_w^2}{K}\ln\left(\frac{r_o}{r}\right)}$$

<u>SPECIAL CASE : $I = 0$</u>

$$h^2 = h_o^2 - \frac{Q}{\pi K}\ln\left(\frac{r_o}{r}\right)$$

IV. ADDITIONAL RESULTS

1. Head at well

Evaluating the solution for h^2 at $r = r_w$:

$$h_w^2 = h_o^2 + \frac{I}{2K}(r_o^2 - r_w^2) - \frac{Q}{\pi K}\ln\left(\frac{r_o}{r_w}\right) - \frac{I r_w^2}{K}\ln\left(\frac{r_o}{r_w}\right)$$

For $I = 0$: $h_w^2 = h_o^2 - \frac{Q}{\pi K}\ln\left(\frac{r_o}{r_w}\right)$

2. Discharge at well

If we know the head at the well and at the outside boundary we can use the above solution to compute the discharge.

$$Q = \frac{\pi K}{\ln\left(\frac{r_o}{r_w}\right)}\left[(h_o^2 - h_w^2) + \frac{I}{2K}(r_o^2 - r_w^2) - \frac{I r_w^2}{K}\ln\left(\frac{r_o}{r_w}\right)\right]$$

3. LOCATION OF THE GROUNDWATER DIVIDE

The extrema of the head solution occur at $\frac{dh}{dr} = 0$.
Since $\frac{dh}{dr} = \frac{1}{2h}\frac{dh^2}{dr}$, the extrema of the head solution also occur at $\frac{dh^2}{dr} = 0$.

For the special case of $I = 0$, the extrema occur at:

$$\frac{d}{dr}\left[h_o^2 - \frac{Q}{\pi K}\ln\left(\frac{r_e}{r}\right)\right] = 0$$

$$\rightarrow \quad -\frac{Q}{\pi K}\left(\frac{r}{r_o}\right)\left(-\frac{r_o}{r^2}\right) = 0$$

Simplifying:

$$\frac{Q}{\pi K}\frac{1}{r} = 0$$

This expression does not yield any extrema. We will have to identify the extrema from a physical argument.

For $Q > 0$ [EXTRACTION]: $h_{MIN} = h_w$, $h_{MAX} = h_o$
For $Q < 0$ [INJECTION]: $h_{min} = h_o$, $h_{max} = h_w$

– Location of h_{max} for the general case of $I \neq 0$

Find where slope=0 for the head solution.

Recall (Flow to a Single Well with Uniform Recharge) head solution:

$$h^2 = h_0^{\ 2} + \frac{I}{2K}(r_0^{\ 2} - r^2) - \frac{Q}{\pi K}\ln\left(\frac{r_0}{r}\right) - \frac{Ir_w^{\ 2}}{K}\ln\left(\frac{r_0}{r}\right)$$

$$h = \sqrt{h_0^{\ 2} + \frac{I}{2K}(r_0^{\ 2} - r^2) - \frac{Q}{\pi K}\ln\left(\frac{r_0}{r}\right) - \frac{Ir_w^{\ 2}}{K}\ln\left(\frac{r_0}{r}\right)}$$

Calculate $\frac{dh}{dr}$ =0:

$$\frac{dh}{dr} = \frac{1}{4}\frac{-\frac{4Ir}{K} + \frac{4Q}{\pi K r} + \frac{4Ir_w^{\ 2}}{Kr}}{\sqrt{4h_0^{\ 2} + \frac{2I}{K}(r_0^{\ 2} - r^2) - \frac{4Q}{\pi K}\ln\left(\frac{r_0}{r}\right) - \frac{4Ir_w^{\ 2}}{K}\ln\left(\frac{r_0}{r}\right)}}$$

Isolate expression for r:

$$r = \frac{\sqrt{(QK + Ir_w^{\ 2}\pi K)I\pi K}}{I\pi K} = \sqrt{r_w^{\ 2} + \frac{Q}{I\pi}}$$

Note: If the value of I equals 0, the equation will not be computable; the location of $h_{\max}$ in such a case, will be at R.

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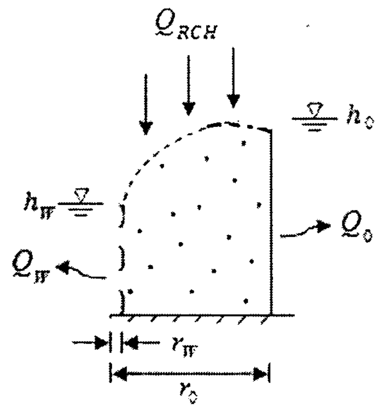
4. Flow Balance



$$Q_{RCH} = Q_w + Q_0$$

$$\pi({r_0}^2 - {r_w}^2)I = Q_w + q_0 A$$

$$\pi({r_0}^2 - {r_w}^2)I = Q_w - K\left.\frac{dh}{dr}\right|_{r_0} 2\pi r_0 h_0$$

$$\pi({r_0}^2 - {r_w}^2)I = Q_w - K\left.\left(\frac{1}{2h}\frac{dh^2}{dr}\right)\right|_{r_0} 2\pi r_0 h_0$$

$$\pi({r_0}^2 - {r_w}^2)I = Q_w - \pi K r_0 \left.\frac{dh^2}{dr}\right|_{r_0}$$

$\left.\frac{dh^2}{dr}\right|_{r_0} = ?$

$$h^2 = {h_0}^2 + \frac{I}{2K}({r_0}^2 - r^2) - \frac{Q}{\pi K} ln\left(\frac{r_0}{r}\right) - \frac{I{r_w}^2}{K} ln\left(\frac{r_0}{r}\right)$$

$$\frac{dh^2}{dr} = -\frac{I}{K}r + \frac{Q}{\pi K}\frac{1}{r} + \frac{I{r_w}^2}{K}\frac{1}{r}$$

$$\therefore \left.\frac{dh^2}{dr}\right|_{r_0} = -\frac{I}{K}r_0 + \frac{Q}{\pi K}\frac{1}{r_0} + \frac{I{r_w}^2}{K}\frac{1}{r_0}$$

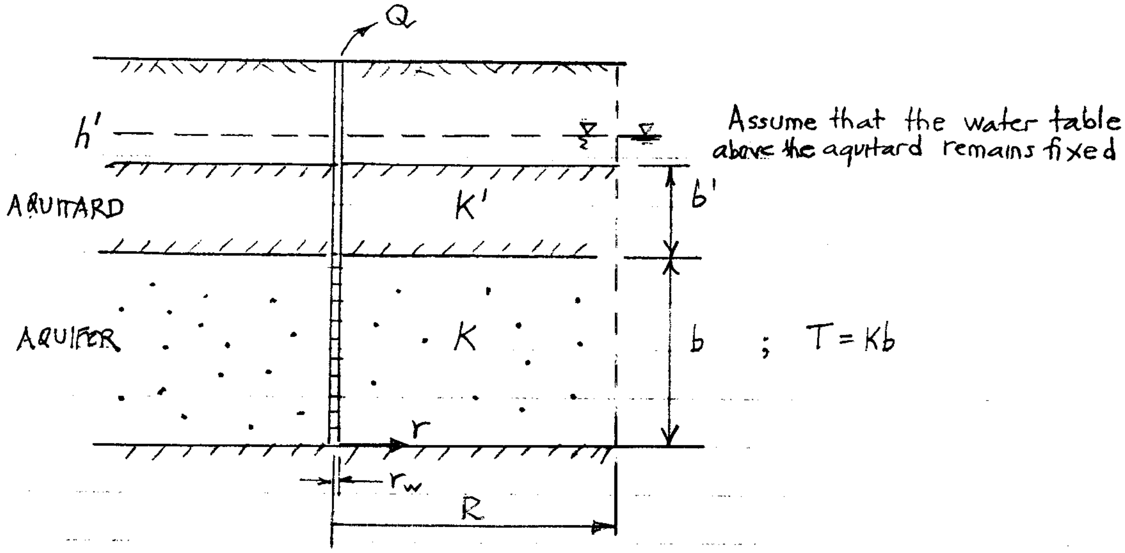
Flow Balance:

$$\pi({r_0}^2 - {r_w}^2)I = Q_w - \pi K r_0\left(-\frac{I}{K}r_0 + \frac{Q}{\pi K}\frac{1}{r_0} + \frac{I{r_w}^2}{K}\frac{1}{r_0}\right)$$

MODEL 10:

STEADY RADIAL FLOW TO A WELL OVERLAIN BY A LEAKY AQUITARD

see Bear (1979) S. 8-4 (p. 312)



1. GOVERNING EQUATION FOR AN AQUIFER OF FINITE RADIAL EXTENT

$$Kb \frac{1}{r}\frac{d}{dr}\left(r\frac{dh}{dr}\right) + \frac{K'}{b'}(h'-h) = 0 \quad ; \quad r_w \le r \le R$$

→ Notice how the governing equation differs in a fundamental way from that for an unconfined aquifer with recharge. For a confined aquifer the recharge (ie., leakage) is a function of the head in the aquifer. This is an "on-demand" source.

2. GENERAL SOLUTION

1) Make the change of variables : $s = h' - h$

$$\therefore \quad \frac{dh}{dr} = -\frac{ds}{dr}$$

The governing equation becomes :

$$-Kb\,\frac{1}{r}\frac{d}{dr}\left(r\frac{ds}{dr}\right) + \frac{K'}{b'}s = 0$$

Expanding and re-arranging the governing equation :

$$\boxed{\frac{d^2s}{dr^2} + \frac{1}{r}\frac{ds}{dr} - \frac{K'}{b'Kb}s = 0} \qquad r_w \le r \le R$$

2) The general solution is :

$$\boxed{s = A I_o(kr) + B K_o(kr)} \quad ; \text{ valid for } k \neq 0$$

I_o : Modified Bessel function of first kind, order zero
K_o : Modified Bessel function of second kind, order zero

where $k = \left(\frac{K'}{b'Kb}\right)^{1/2} \equiv \left(\frac{K'}{b'T}\right)^{1/2} = \frac{1}{\lambda}$

$$[k] = \left(\frac{LT^{-1}}{L\,LT^{-1}L}\right)^{1/2} = L^{-1}$$

3. PARTICULAR SOLUTION FOR A DRAWDOWN-CONTROLLED WELL

For a drawdown-controlled well, the boundary conditions are:

$$h(r_w) = h_w$$
$$h(R) = h'$$

← The bounding head is assumed to be the same as the head at the top of the aquitard.

or, in terms of drawdowns:

$$s(r_w) = s_w = h' - h_w$$
$$s(R) = 0$$

Recall the general solution:

$$s = A I_0(kr) + B K_0(kr)$$

where: $k = \left(\frac{K'}{b' K b} \right)^{1/2}$

The coefficients A and B are evaluated by considering the boundary conditions.

i) $r = r_w$:

$$s(r_w) = s_w = A I_0(kr_w) + B K_0(kr_w)$$

ii) $r = R$:

$$s(R) = A I_0(kR) + B K_0(kR) = 0$$

$$\hookrightarrow \quad B = -A \frac{I_0(kR)}{K_0(kR)}$$

Substituting for B in the first boundary condition:

$$A I_0(kr_w) + \left(-A \frac{I_0(kR)}{K_0(kR)}\right) K_0(kr_w) = s_w$$

$$\therefore \quad A = \frac{s_w}{I_0(kr_w) - \frac{I_0(kR)}{K_0(kR)} K_0(kr_w)}$$

Re-arranging slightly:

$$A = s_w \frac{K_o(kR)}{I_o(kr_w)K_o(kR) - I_o(kR)K_o(kr_w)}$$

$$\therefore \quad B = -s_w \frac{K_o(kR)}{I_o(kr_w)K_o(kR) - I_o(kR)K_o(kr_w)} \; \frac{I_o(kR)}{K_o(kR)}$$

Substituting for A and B in the general solution yields:

$$s = s_w \frac{K_o(kR)}{I_o(kr_w)K_o(kR) - I_o(kR)K_o(kr_w)} \; I_o(kr)$$

$$- s_w \frac{I_o(kR)}{I_o(kr_w)K_o(kR) - I_o(kR)K_o(kr_w)} \; K_o(kr)$$

$$\therefore \quad s = s_w \frac{K_o(kR)I_o(kr) - I_o(kR)K_o(kr)}{I_o(kr_w)K_o(kR) - I_o(kR)K_o(kr_w)}$$

or

$$\boxed{s = s_w \frac{I_o(kr)K_o(kR) - I_o(kR)K_o(kr)}{I_o(kr_w)K_o(kR) - I_o(kR)K_o(kr_w)}}$$

CHECK :

(i) $r = r_w$:

$$s = s_w \cdot \frac{I_o(kr_w)\,K_o(kR) - I_o(kR)\,K_o(kr_w)}{I_o(kr_w)\,K_o(kR) - I_o(kR)\,K_o(kr_w)}$$

$$= s_w \quad \checkmark$$

(ii) $r = R$:

$$s = s_w \cdot \frac{I_o(kR)\,K_o(kR) - I_o(kR)\,K_o(kR)}{I_o(kr_w)\,K_o(kR) - I_o(kR)\,K_o(kr_w)}$$

$$= 0 \quad \checkmark$$

4. SOLUTION FOR PUMPING RATE, Q:

$$Q = -2\pi r K b \left.\frac{dh}{dr}\right|_{r_w} = +2\pi r K b \left.\frac{ds}{dr}\right|_{r_w}$$

$$= +2\pi r K b \frac{d}{dr}\left[s_w \cdot \frac{I_0(kr)K_0(kR) - I_0(kR)K_0(kr)}{I_0(kr_w)K_0(kR) - I_0(kR)K_0(kr_w)} \right]_{r=r_w}$$

$$= +2\pi r_w K b\, s_w \left.\frac{kI_1(kr)K_0(kR) + I_0(kR)\,k\,K_1(kr)}{I_0(kr_w)K_0(kR) - I_0(kR)K_0(kr_w)}\right|_{r=r_w}$$

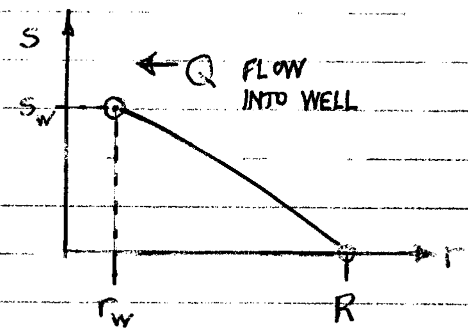
$$= +2\pi r_w K b\, s_w \frac{kI_1(kr_w)K_0(kR) + kI_0(kR)K_1(kr_w)}{I_0(kr_w)K_0(kR) - I_0(kR)K_0(kr_w)}$$

$$\therefore \boxed{Q = +2\pi r_w K b\, s_w \cdot k \frac{I_1(kr_w)K_0(kR) + I_0(kR)K_1(kr_w)}{I_0(kr_w)K_0(kR) - I_0(kR)K_0(kr_w)}}$$

The denominator is negative

SIGN CONVENTION FOR Q:

For positive s_w, the predicted flow rate is negative.



→ A negative flow rate corresponds to flow in the negative r direction; that is flow toward the well. This make sense.

APPENDIX C

Notes on analytical solutions for groundwater inflow to the base of a circular excavation:

Model 11. Forchheimer (1914) solution

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1. Conceptual model

The conceptual model for the Forchheimer (1914; p. 75) solution is shown in Figure 1. The solid line in the impervious layer represents the potentiometric surface in the underlying aquifer. Implicit in the conceptual model is that the source of water is a constant-head surface at some distance $x >> r_w$, and that the aquifer is thick. Hvorslev (1951) adopted the Forchheimer (1914) as his shape factor Case 3 (p. 31) and Case B (p. 44).

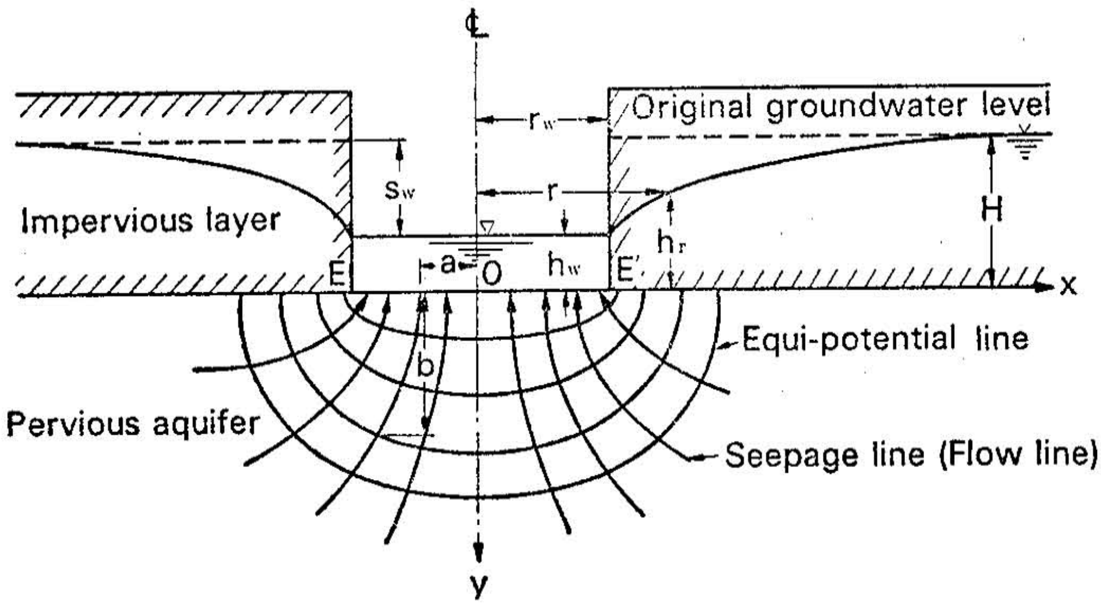

Figure 1. Conceptual model for the Forchheimer (1914) solution

2. Solutions for head and discharge

The solution for the hydraulic head in the confined aquifer is (Suzuki and Yokoya, Eq 5):

$$H - h(r) = \frac{Q}{2\pi K r_w} \sin^{-1}\left(\frac{r_w}{r}\right)$$

The solution for the inflow to the excavation is:

$$Q = 4Kr_w s_w$$

<u>Check</u>:

In the limit as $r >> r_w$, the solution reduces to:

$$H - h(r \gg r_s) \rightarrow \frac{Q}{2\pi K r_w} sin^{-1}(0) = 0$$

That is,

$$h(r \gg r_s) \rightarrow H \; \surd$$

Given the drawdown in the excavation, $s_w = H\text{-}h_w$, the discharge is given by:

$$\begin{aligned} H - h_w &= \frac{Q}{2\pi K r_w} \sin^{-1}(1) \\ &= \frac{Q}{2\pi K r_w} \frac{\pi}{2} \\ &= \frac{Q}{4Kr_w} \end{aligned}$$

Re-arranging:

$$Q = 4Kr_w s_w$$

This is Suzuki and Yokoya, Eq 1.

3. Calculation of radius of influence

The head at a radial distance *R* is obtained by evaluating the head solution at *r* = *R*:

$$H - h_R = \frac{Q}{2\pi K r_w} \sin^{-1}\left(\frac{r_w}{R}\right)$$

Solving for *R* yields:

$$\sin\left[\left(H - h_R\right)\frac{2\pi K r_w}{Q}\right] = \frac{r_w}{R}$$

$$\rightarrow R = \frac{r_w}{\sin\left[\left(H - h_R\right)\frac{2\pi K r_w}{Q}\right]}$$

This is Suzuki and Yokoya, Eq 6.

Substituting for the discharge in the expression for *Q* yields:

$$R = \frac{r_w}{\sin\left[\left(H - h_R\right)\frac{2\pi K r_w}{\left[4 K r_w s_w\right]}\right]}$$

$$= \frac{r_w}{\sin\left[\left(H - h_R\right)\frac{\pi}{\left[2 s_w\right]}\right]} = \frac{r_w}{\sin\left[\frac{\pi}{2}\frac{\left(H - h_R\right)}{s_w}\right]}$$

Defining the drawdown at *r* = *R* as s_R, we can write the expression for the radius of influence as.

$$\frac{R}{r_w} = \frac{1}{\sin\left[\frac{\pi}{2}\left(\frac{s_R}{s_w}\right)\right]}$$

This expression is Suzuki and Yokoya Eq 7.

Estimation of the radius of influence

The relation between the radius of influence and the drawdown is plotted in Figure 2.

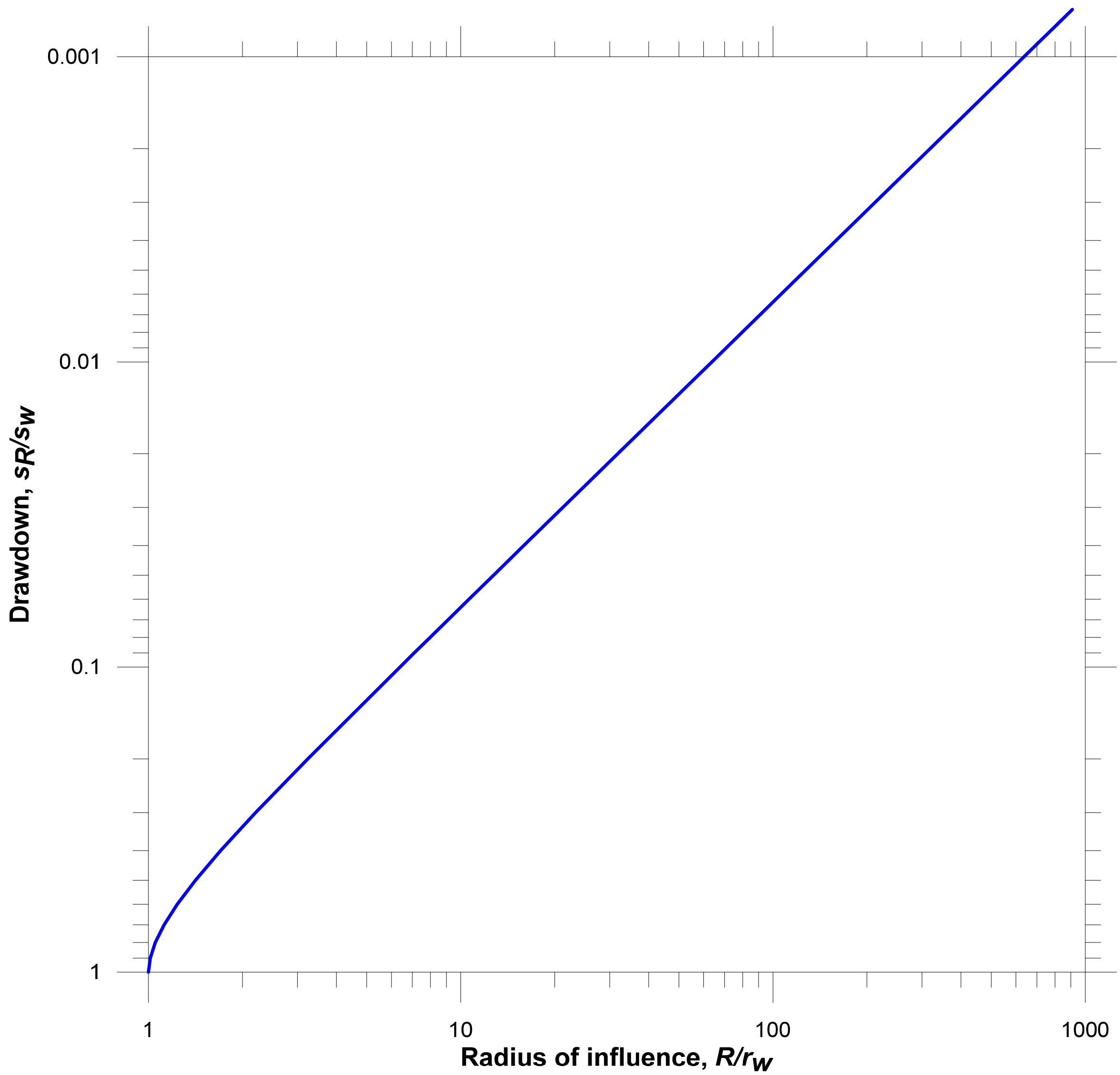

Figure 2. Radius of influence for the Forchheimer solution

The radius of influence is estimated in three steps:

1. Choose a criterion for negligible drawdown, expressed as a fraction of the drawdown in the excavation;
2. Estimate R/r_w from the chart; and
3. Multiple R/r_w by the effective radius of the excavation, r_w.

<u>Example</u>:

- Assume the drawdown in the excavation is 10 m
- Assume that a "negligible" drawdown is 1.0 cm
- Calculate s_R/s_w = 1.0 cm/10 m = 0.001
- From the chart: R/r_w = 636
- If r_w = 50 m, R = 31,800 m (!)

References

Forchheimer, P., 1914: **Hydraulik**, B.G. Teubner, Leipzig and Berlin, p. 439

Hvorslev, M.J., 1951: Time Lag and Soil Permeability in Ground-Water Observations, Bulletin No. 36, Waterways Experiment Station, U.S. Army Corps of Engineers, Vicksburg, Mississippi, 50 p.

Suzuki, O., and H. Yokoya, 1992: Application of Forchheimer's formula to dewatered excavation as a large circular well, *Soils and Foundations*, vol. 32, no. 1, pp. 215-221.

Notes on analytical solutions for groundwater inflow to the base of a circular excavation:

Model 12. Hvorslev (1951) Case 4/C model [Harza/Taylor]

Hvorslev (1951) cited "empirical data" by Harza and Taylor as the sources for this solution, which Hvorslev designated Case 4 (p. 31) and Case C (p. 44) ["soil flush with bottom in uniform soil"]:

$$Q = 5.50\, K h_t\, r_0$$

Taylor (1948) obtained his result from the carefully drawn flownet reproduced in Figure 1.

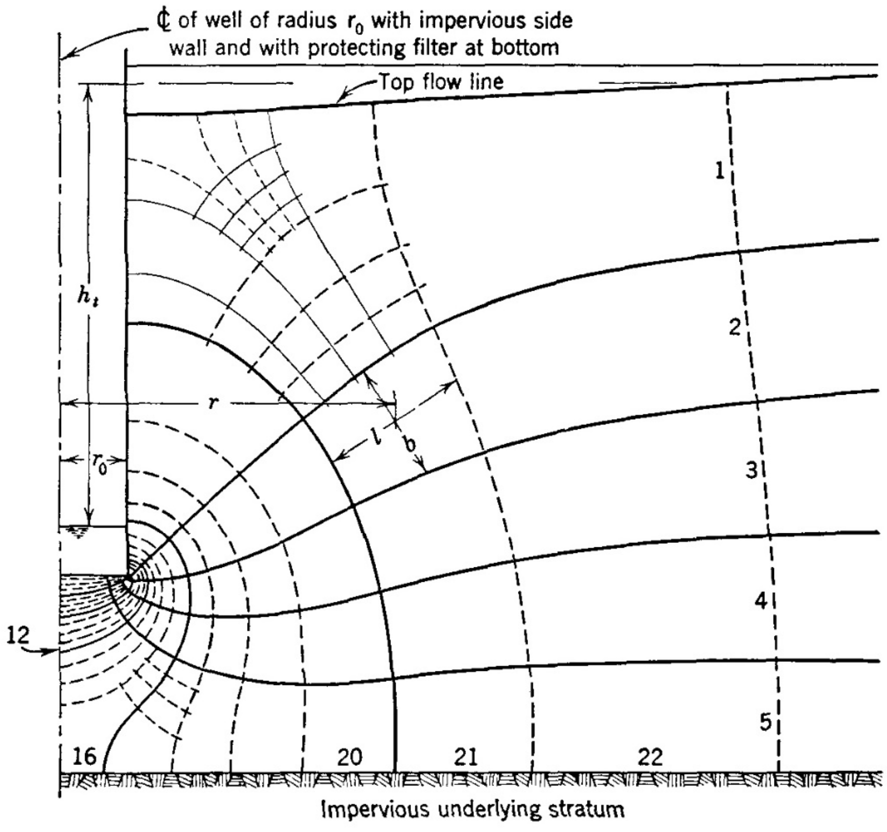

Figure 1. Flow net for the radial flow to the bottom of an excavation
(Reproduced from Taylor, 1948)

The Taylor flow net solution for the inflow into the base can be written as:

$$Q = 2\pi K h_t \frac{n_f}{n_d}\left(\frac{rb}{l}\right)$$

<u>Parameters</u>:

r: 5.1 r_0
b: 1.8 r_0
l: 2.3 r_0

From the flow net solution:

$n_f = 5$
$n_d = 22$

$$Q = 2\pi K h_t \frac{(5)}{(22)}\left(\frac{(5.1\, r_0)(1.8\, r_0)}{(2.3\, r_0)}\right) = 5.70\, K h_t\, r_0$$

Taylor also referred to the additional investigations of Harza (1935) conducted with electric analog methods. Taylor indicated that the Harza results confirmed that the inflow is insensitive to the radial extent of the cross section, the depth below the well, and the depth of the soil. This is because practically all head is lost near the entrance to the well.

Silvestri and others (2012) developed an exact analytical solution for the problem of an infinitely thick aquifer:

$$Q = 2.804\, K\Delta H D = \; 5.608\, K\Delta H\, r_0$$

This solution is amazing close to Taylor's result, demonstrating the power of a well-constructed flownet. The agreement also confirms Taylor's indication that the assumption regarding the thickness of the aquifer is not important, as so much of the head loss occurs right around the entrance of the well.

References

Harza, L.F., 1935: Uplift and seepage under dams, *Transactions of the American Society of Civil Engineers*, vol. 100, pp. 1352-1385.

Hvorslev, M.J., 1951: Time Lag and Soil Permeability in Ground-Water Observations, Bulletin No. 36, Waterways Experiment Station, U.S. Army Corps of Engineers, Vicksburg, Mississippi, 50 p.

Silvestri, V., G. Abou-Samra, and C. Bravo-Jonard, 2012: Shape factors for cylindrical piezometers in uniform soil, *Ground Water*, vol. 50, no. 2, pp. 279-284.

Taylor, D.W., 1948: **Fundamentals of Soil Mechanics**, John Wiley & Sons, New York, New York.