Analytical solutions for the estimation of long-term rates of lateral groundwater inflow into rectangular excavations

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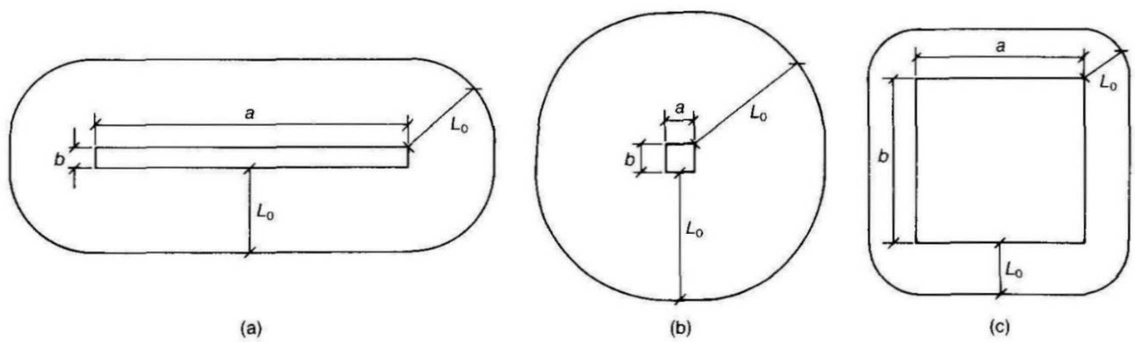
Overview

There are no closed-form analytical solutions for flow into the sides of rectangular excavations. Based on the work of Powrie and Preene (1992) approximate approaches for estimating the steady groundwater flow into rectangular excavations in confined aquifers are developed for three cases:

1. Flow into a "long" excavation;
2. Flow into an approximately square excavation with a distant recharge boundary; and
3. Flow into an approximately square excavation with a nearby recharge boundary.

These cases are shown schematically below. In addition to the simplified analyses, the general "shape factor" approach of Powrie and Preene (1992) is presented and assessed.



Conceptual models for groundwater flow into rectangular excavations
Reproduced from Powrie and Preene (1992)

Contents

1. Analytical solution for flow into a "long" excavation
2. Analytical solution for flow into an approximately square excavation with a distant recharge boundary
3. Analytical solution for flow into an approximately square excavation with a nearby recharge boundary
4. General analysis of flow into an excavation of any dimensions and distance to the recharge boundary
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Example analyses comparing the results of the analytical approaches with the results of numerical models are presented in a separate document appended to these notes.

1. Flow into a "long" excavation

The conceptual model for a long excavation (*a* >> *b*) is shown schematically in Figure 1.

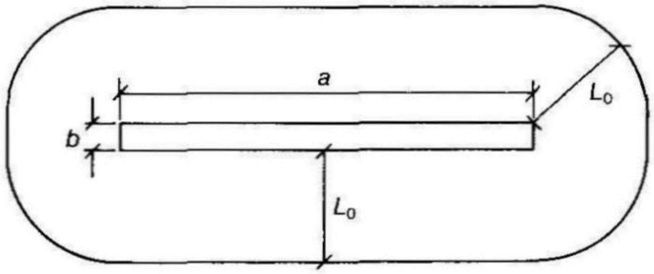


Figure 1. Conceptual model for a "long" excavation

An approximate expression for the steady inflow to the excavation is developed following an approach presented in Driscoll (1986; Groundwater and Wells, 2nd edition, p. 740). The flow rate into the excavation is approximated as:

Q ≈2 × linear flow into sides + 2 × radial flow into each end semi-circle

Designating the flow into a side as Q_S and the flow into an end as Q_E, the total flow into the excavation is:

$$Q = 2Q_S + 2Q_E$$

The expressions for Q_S and Q_E are developed on the following pages. Adding the expressions yields:

$$Q = 2\left[KD\frac{(H-h_{ex})}{L_o}a\right] + 2\left[\pi KD\frac{(H-h_{ex})}{\ln\left\{\frac{2L_0}{b}\right\}}\right]$$

Here K is the horizontal hydraulic conductivity and D is the aquifer thickness. H is the head at the recharge boundary and h_{ex} is the head in the excavation.

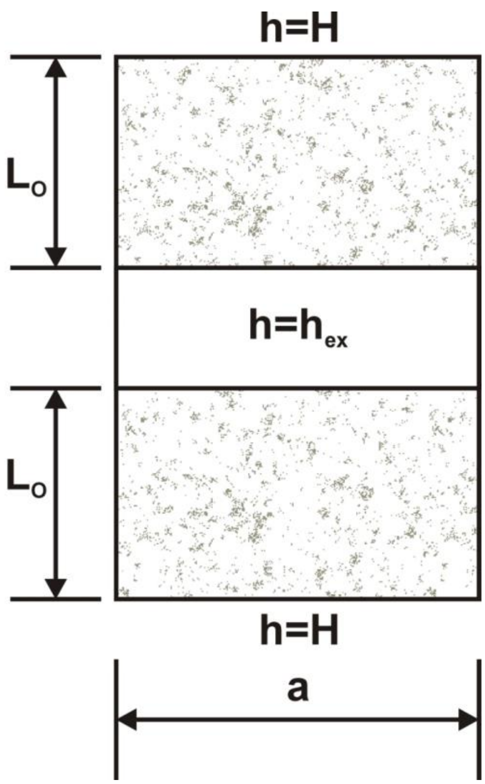
Simplifying yields:

$$Q = 2KD\,(H-h_{ex})\left[\frac{a}{L_o} + \frac{\pi}{\ln\left\{\frac{2L_o}{b}\right\}}\right]$$

This solution is identical to Powrie and Preene (1992; Equation [1]). It can also be shown that the solution is the confined flow equivalent to the solution for unconfined flow presented in Driscoll (1986; p. 741).

Development of the solution for linear flow into the long sides of the excavation

The conceptual model for flow into the long side is shown below.



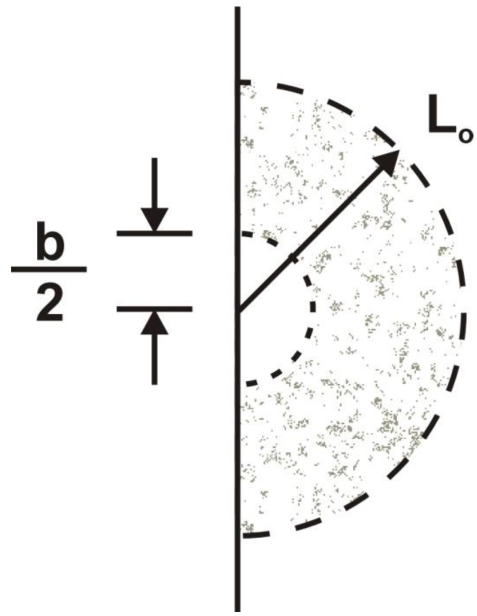
The inflow into each of the long sides of the excavation is approximated as:

$$Q_S = KD\frac{(H - h_{ex})}{L_o}\ a$$

Here *H* denotes the groundwater level at a linear distance L_0.

Development of the solution for radial flow into the ends of the excavation

The conceptual model for the flow into each end of the excavation is shown below.



The inflow into each end of the excavation is approximated from the Thiem solution for steady radial flow:

$$Q_E = \frac{1}{2}\left[2\pi KD \frac{(H - h_{ex})}{\ln\left\{\frac{L_0}{\left(\frac{b}{2}\right)}\right\}}\right]$$

Here H denotes the groundwater level at a radial distance L_0.

The solution for the flow into each end of the excavation can be written in a simpler form as:

$$Q_E = \pi KD \frac{(H - h_{ex})}{\ln\left\{\frac{2L_0}{b}\right\}}$$

Powrie and Preene (1992) compared the results of the simplified solution for a long excavation against the results from finite element simulations. As shown in Figure 2, over the range of 0.01 < L_0/a < 1, the results from the solution approximate the numerical results closely (within about ±20%).

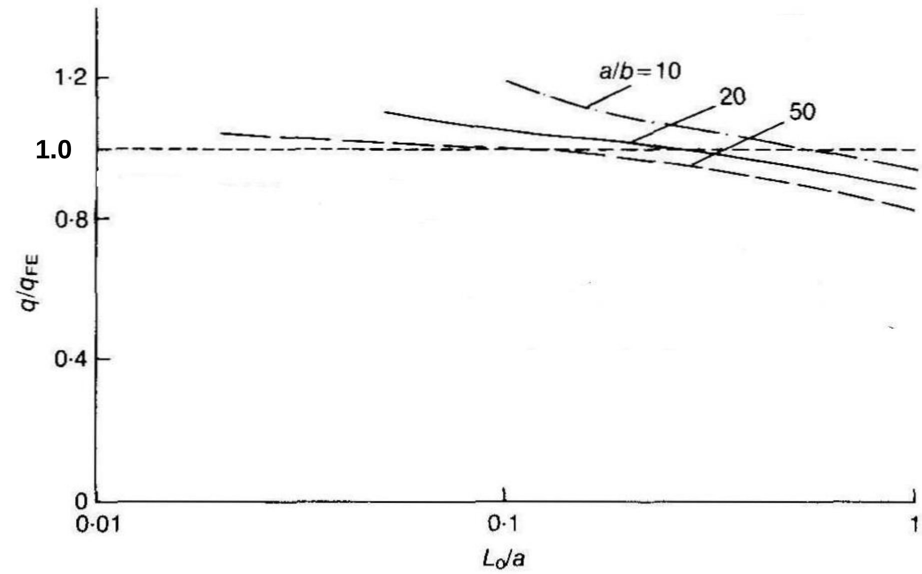


Figure 2. Comparison of results for a long excavation
Reproduced from Powrie and Preene (1992)

2. Flow into an "approximately square" excavation ($a \cong b$) with a distant recharge boundary

The case of an "approximately square" excavation with a recharge boundary located at a distance that is much greater than the dimensions of the excavation is shown in illustrated in Figure 3.

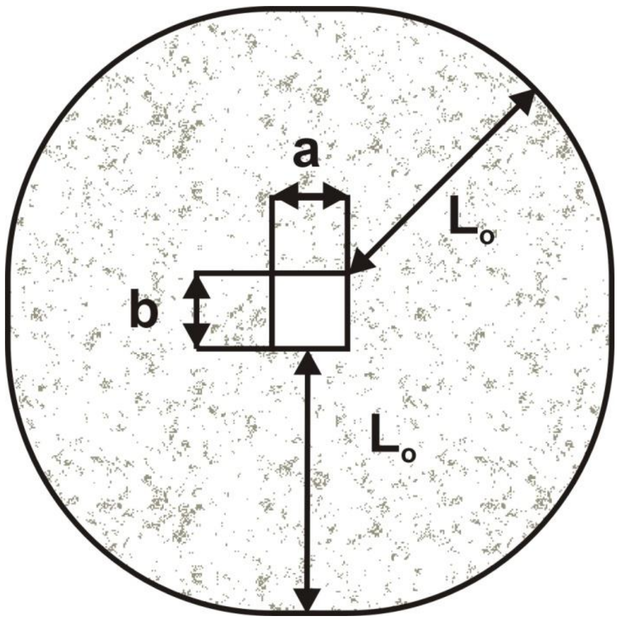


Figure 3. Conceptual model for an "approximately square" excavation, ($a \cong b$) and a distant boundary (L_0 >> a, b)

Following the approach of Mansur and Kaufman (1962), the inflow to the excavation is estimated assuming radial flow to an equivalent well:

$$Q = \frac{2\pi KD\ (H - h_{ex})}{\ln\left\{\frac{L_o}{r_{eq}}\right\}}$$

The term r_{eq} represents the equivalent radius of the excavation.

Specification of the equivalent radius of the excavation

Three approaches for specifying the equivalent radius are discussed below. Comparisons made by Powrie and Preene (1992) suggest that neither approach for calculating r_{eq} is universally superior.

2.1 Approach #1: Equivalent radius defined on the basis of the equivalent area

For a circular excavation that has the same plan area as the actual rectangular excavation:

$$\pi r_{w-eq\_A}^{2} = ab$$

The equivalent radius is therefore:

$$r_{w-eq\_A} = \sqrt{\frac{ab}{\pi}}$$

This is Powers and others (2007; Equation [6.8]), and Cashman and Preene (2013; Equation [7.2].

Powrie and Preene`s comparison of the results of finite element simulations and those obtained with the equivalent well solution based on equal area is reproduced in Figure 4. For values of L_0/a greater than about 2, the results from the simplified radial flow analysis are relatively close the numerical simulations (within ±20%).

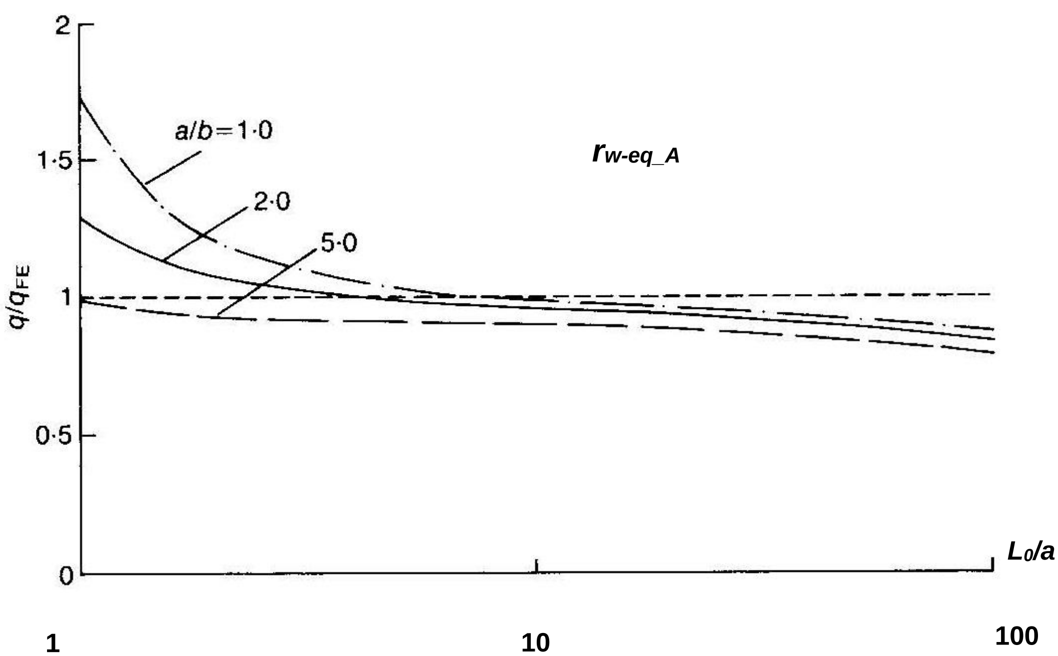


Figure 4. Assessment of equivalent single well solution based on area
(Reproduced from Powrie and Preene, 1992; Figure 4)

2.2 Approach #2: Equivalent radius defined on the basis of the equivalent perimeter

For a circular excavation that has the same plan perimeter as the actual rectangular excavation:

$$2\pi r_{w-eq\_P} = 2(a+b)$$

The equivalent radius is therefore:

$$r_{w-eq\_P} = \frac{a+b}{\pi}$$

This is Powers and others (2007; Equation [6.9]), and Cashman and Preene (2013; Equation [7.1]).

Powrie and Preene`s comparison of the results of finite element simulations and those obtained with the equivalent well solution based on equal perimeter is reproduced in Figure 5. For values of L_0/a greater than about 5, the results from the simplified radial flow analysis are relatively close the numerical simulations (within ±10%). The results shown for the equivalent area and equivalent perimeter suggest that neither approach is universally superior.

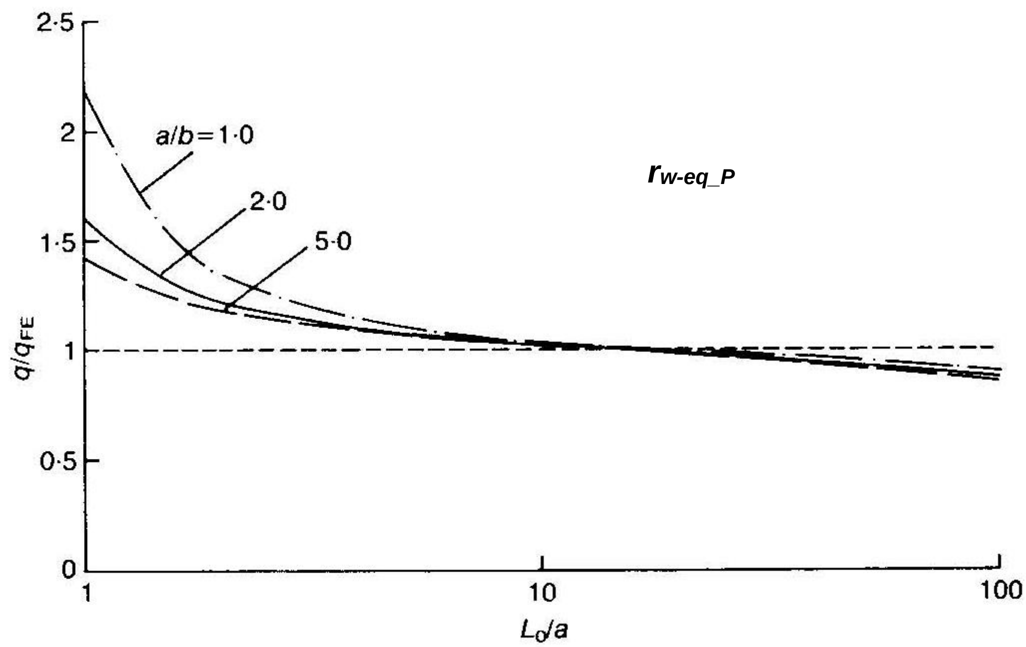


Figure 5. Assessment of equivalent single well solution based on perimeter

2.3 Approach #3: Equivalent radius based on the approach of Mansur and Kaufman (1962)

Mansur and Kaufman (1962) provide the following formula for calculating the equivalent radius of a rectangular excavation (their Equation [39]):

$$r_{w-eq} = \frac{4}{\pi}\sqrt{b_1 b_2}$$

The quantities b_1 and b_2 are the half-length and half-width of the excavation, respectively. Using the present notation:

$$r_{w-eq} = \frac{4}{\pi}\sqrt{\frac{a}{2}\frac{b}{2}} = \frac{2}{\pi}\sqrt{ab}$$

We have not been able to find any details on the development of Mansur and Kaufman's formula for the equivalent radius. However, it can be shown that their formula matches the results of the equivalent-area for a square excavation:

$$r_{w-eq\_A} = \frac{4}{\pi}\sqrt{\frac{a}{2}\frac{a}{2}} = \frac{2}{\pi}\,a$$

3. Flow into an "approximately square" excavation ($a \cong b$) with a nearby recharge boundary

The case of an "approximately square" excavation with a recharge boundary located relatively close to the excavation is shown in Figure 6. For this case, Powrie and Preene (1992) again suggest using the equivalent single well approach; however, they consider the approach to be more approximate than the approach applied for a distant recharge boundary. Powrie and Preene discuss two approximate approaches for estimating the steady flow into the excavation for this case. The two approximate approaches are developed in the following sections.

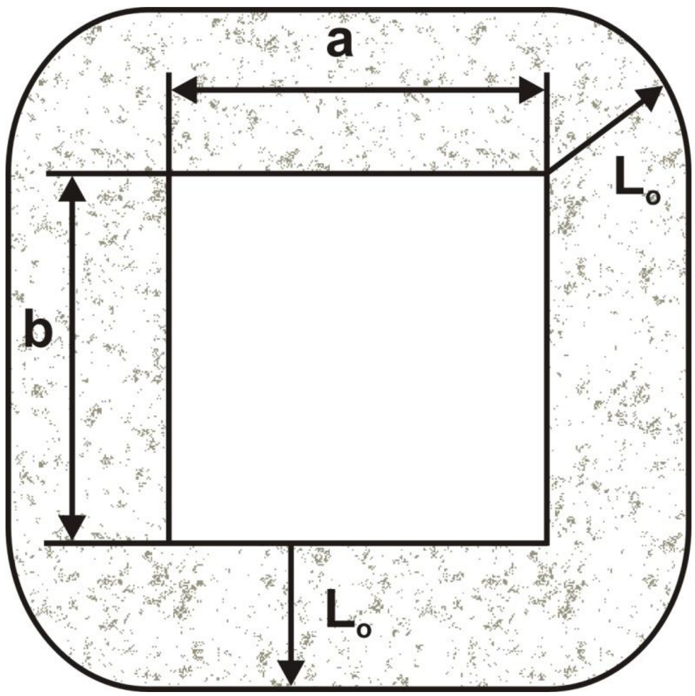


Figure 6. Approximately square excavation ($a \cong b$): nearby boundary (L_0 ~ a, b)

3.1 Approximate analysis #1

The flow rate is estimated by assuming linear flow into the sides of the excavation, as shown below.

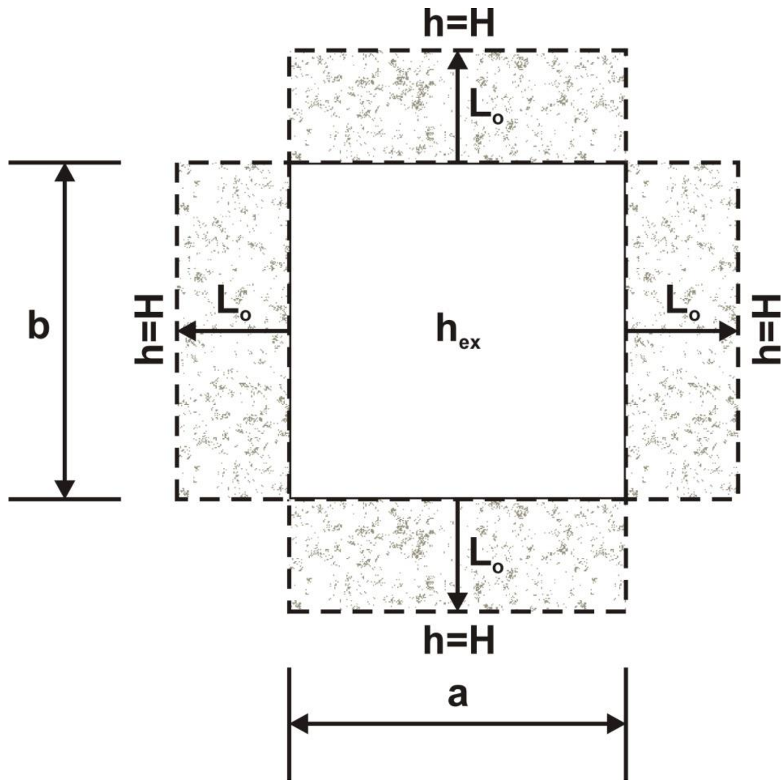


Figure 7. Conceptual model for approximate analysis #1

The inflow to the excavation is approximated as the linear flow into each side of the excavation:

$$Q = 2 * \left[KD\frac{(H - h_{ex})}{L_0}\ a\right] + 2 * \left[KD\frac{(H - h_{ex})}{L_0}\ b\right]$$

Collecting terms:

$$Q = KD\ (H - h_{ex})\frac{2(a+b)}{L_0}$$

In Figure 8 the results of the first approximate solution are compared with finite element results. The results from the first simplified analytical approach are relatively close to the numerical results for values of L_0/a less than 0.1 (within 10%). For values of L_0/a greater than 0.1, the simplified analytical solution underestimates the flows significantly. Powrie and Preene (1992) consider this analysis as *too* approximate, as it neglects flow from the corners.

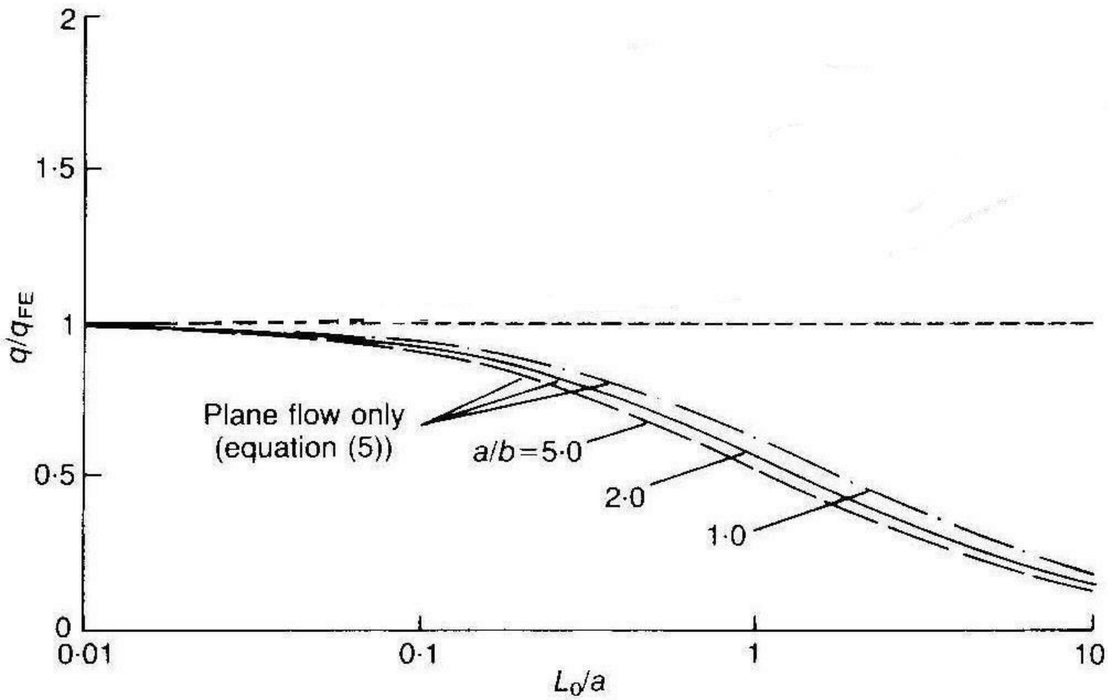


Figure 8. Assessment of simplified solution #1 for a nearby boundary
(Reproduced from Powrie and Preene, 1992; Figure 5)

3.2 Approximate analysis #2

For the second approximate analysis, flow into the corners of the excavation is considered using the area estimated in terms of the average perimeter as defined by Cedergren (1989):

$$Q = K\frac{(H - h_{ex})}{L_0}A$$

$$= K\frac{(H - h_{ex})}{L_0}\bar{P}D$$

The average perimeter is given by:

$$\bar{P} = 2\,[a + b] + 4\left[\frac{1}{4}\,2\pi\frac{L_o}{2}\right]$$

Simplifying:

$$\bar{P} = 2(a + b) + \pi L_o$$

Substituting into the solution for the inflow:

$$Q = KD\,\frac{(H - h_{ex})}{L_0}\cdot[2(a + b) + \pi L_o]$$

Simplifying yields:

$$\boxed{Q = KD(H - h_{ex})\left[\frac{2(a + b)}{L_o} + \pi\right]}$$

The solution for the approximate analysis #2 is identical to Powrie and Preene (1992; Equation 4). The results of the second approximate solution are compared with finite element results in Figure 9. For values of L_0/a less than 0.1, the results match closely. For values of L_0/a greater than 1, the relative errors in the simplified analytical solution increase beyond 20%.

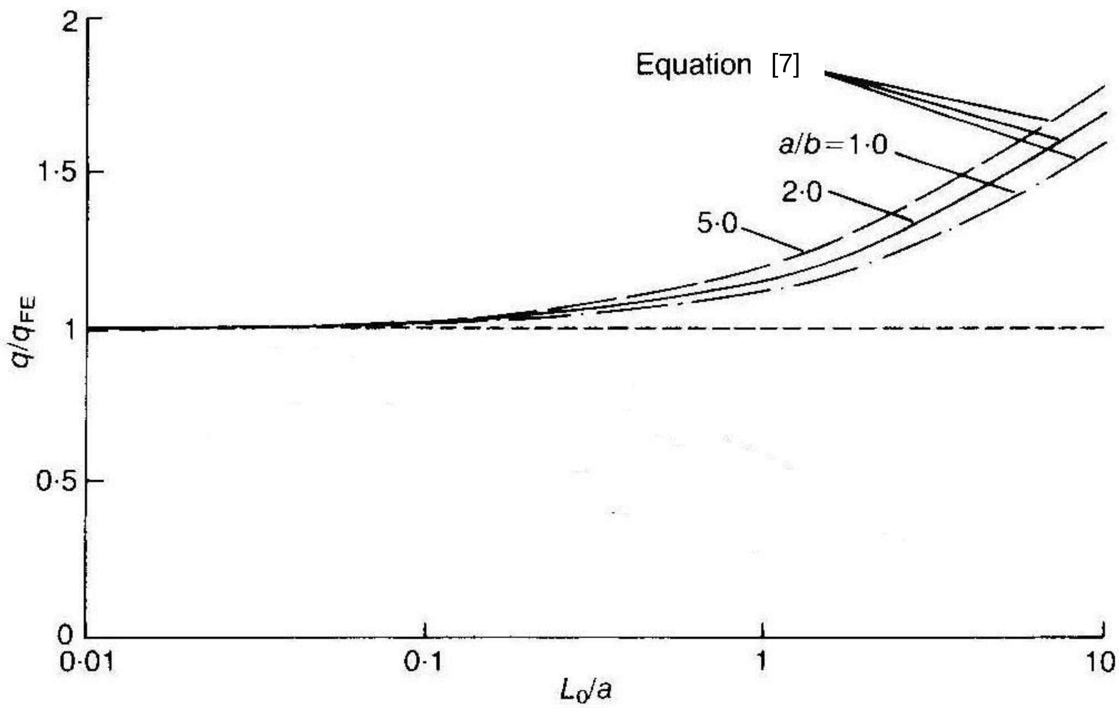


Figure 9. Assessment of the simplified solution #2 for a nearby boundary
(Reproduced from Powrie and Preene, 1992; Figure 5)

4. Flow into an excavation of any dimensions and distance to the recharge boundary

To estimate the inflow to a rectangular excavation of any dimensions, Powrie and Preene (1992) proposed a general relation of the form:

$$Q = KD\ (H - h_{ex})G$$

The parameter *G* is a "shape factor" that depends on the dimensions $\frac{a}{b}$ and $\frac{L_0}{a}$ (or $\frac{L_0}{b}$).

Powrie and Preene (1992) developed values of *G* with the results of finite-element analyses. The plot of *G* is reproduced in Figure 10. In the plot, *q* denotes the inflow, *k* the hydraulic conductivity, *D* the aquifer thickness and $\overline{h_w}$ the head difference between the boundary and the excavation (*H*-h_{ex}).

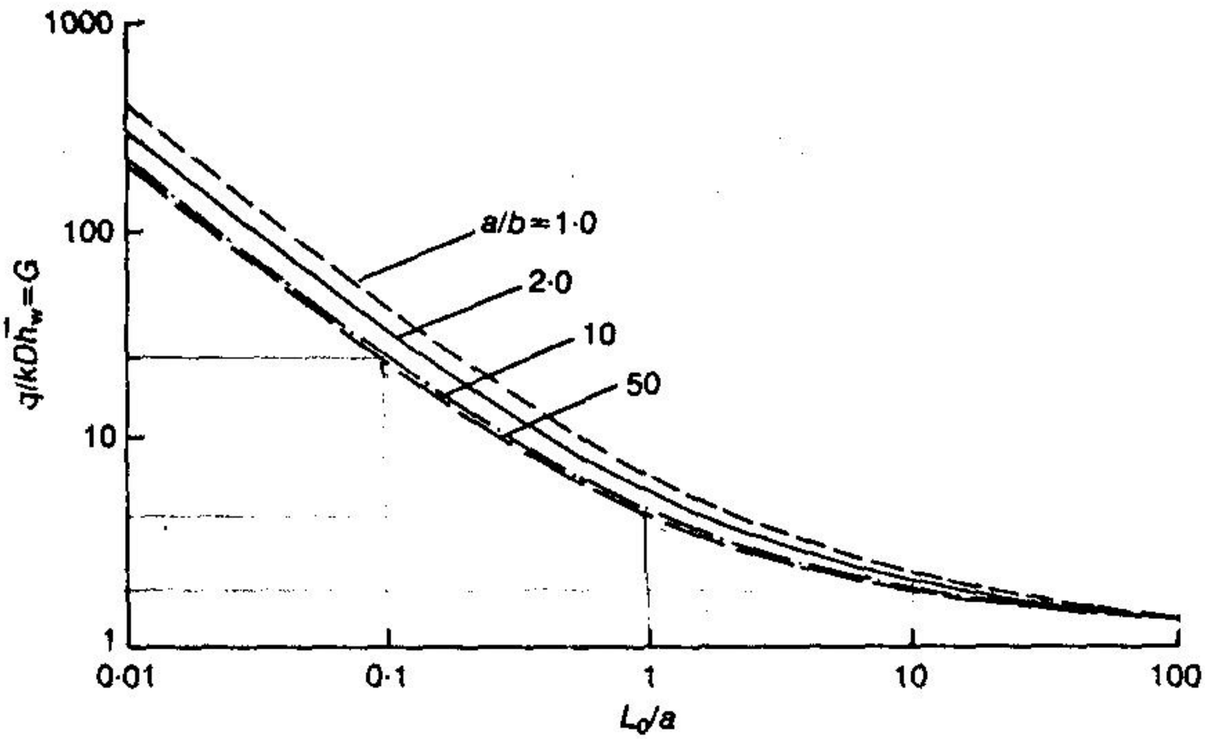


Figure 10. Shape factor *G*
(Reproduced from Powrie and Preene, 1992)

To simplify estimation of values of *G*, the curves plotted in Powrie and Preene (1992) have been digitized and are re-plotted in Figure 11. For values of *a/b* greater than about 10, the values of *G* closely approximate a single line. Jeff Markle has a developed a "calculator" for the *G* factor, which automatically interpolates from the published curves for any values of *a*/*b* and L_0/a.

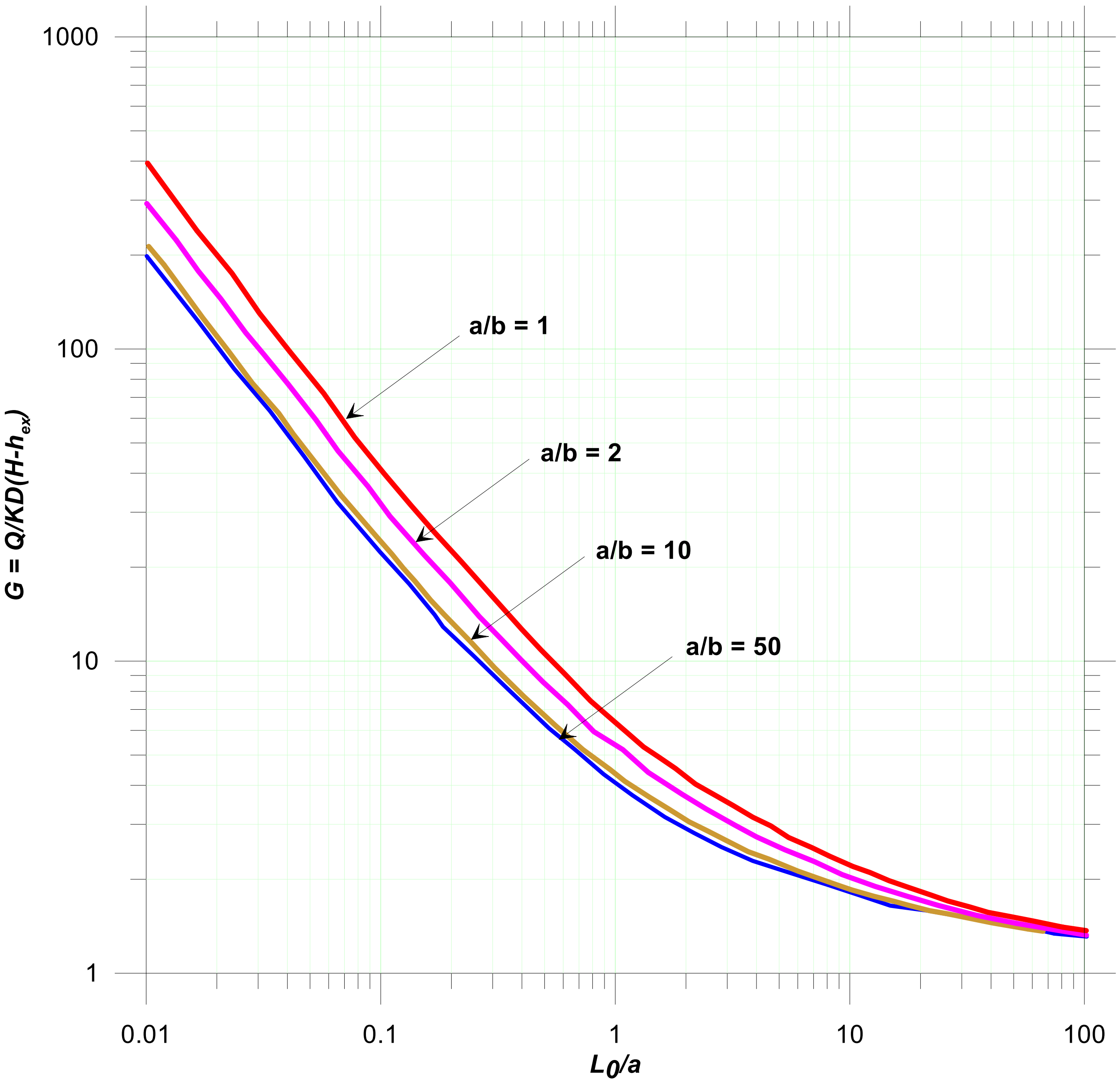


Figure 11. Shape factor *G* re-drafted

Assessment of the Powrie and Preene (1992) shape factor approach

1. The values of Powrie and Preene's shape factor *G* appear to reach asymptotic values for large values of *a/b*. Are the values of *G* for large *a/b* consistent with the simplified results for a "long" excavation?

Recall that for *a*>>*b*, the flow into a "long" excavation is:

$$Q \cong 2KD\,(H - h_{ex})\frac{a}{L_0}$$

Comparing this equation with the general form yields:

$$G_{long} = 2\frac{a}{L_0} \equiv 2\left(\frac{1}{\left(\frac{L_0}{a}\right)}\right)$$

Powrie and Preene's shape factors for the general case are plotted in Figure 11 with the simplified solution for a long excavation. As shown in the figure, the shape factor *G* approaches the appropriate asymptotic value for relatively small values of L_0/a.

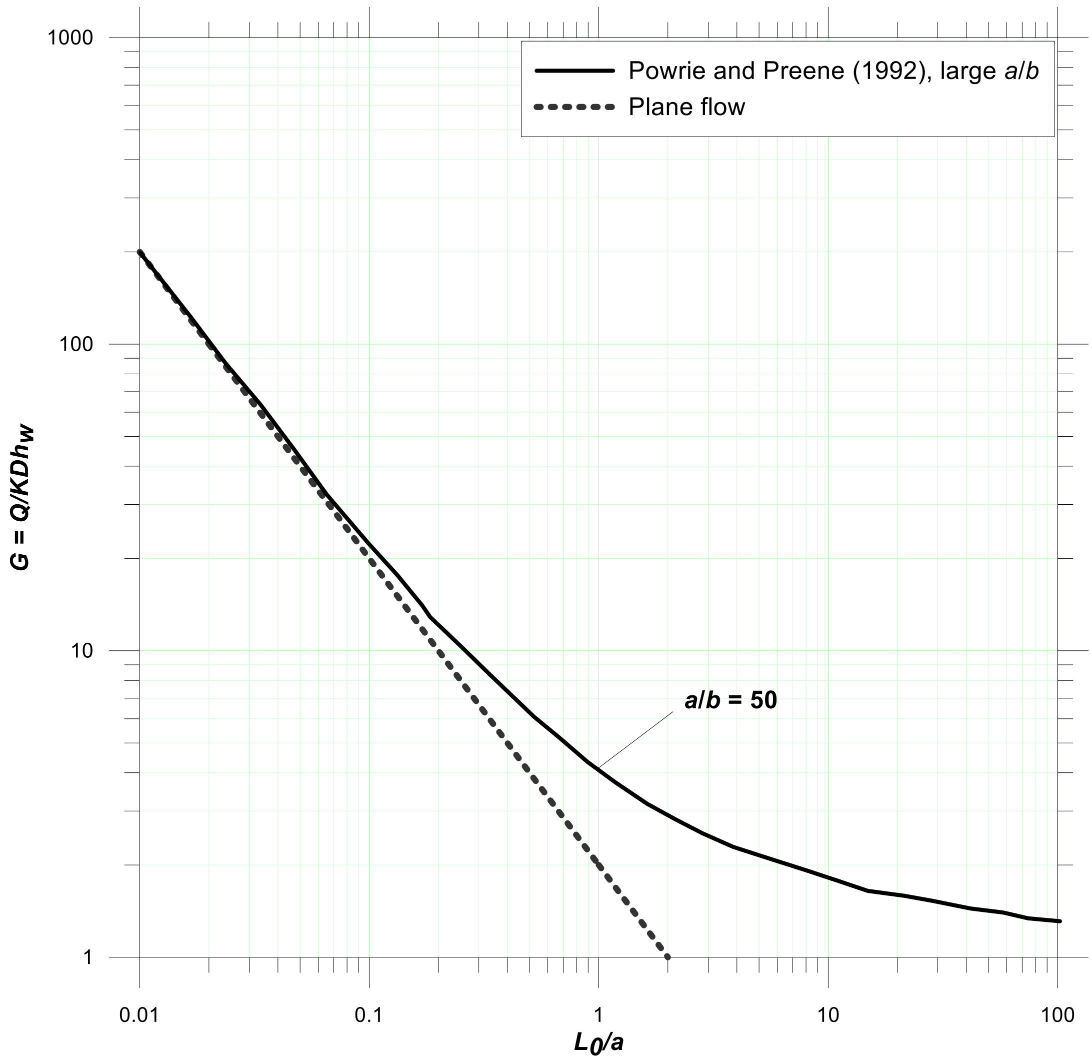


Figure 12. Assessmennt of the asymptotic case of a "long" excavation

2. The values of the shape factor *G* also appear to converge for large values of L_0/a. Are the values of *G* for large values of L_0/a consistent with the simplified results for a “distant” boundary? As suggested in Figure 3, flow should become progressively more radial than linear as L_0 gets much larger than *a*.

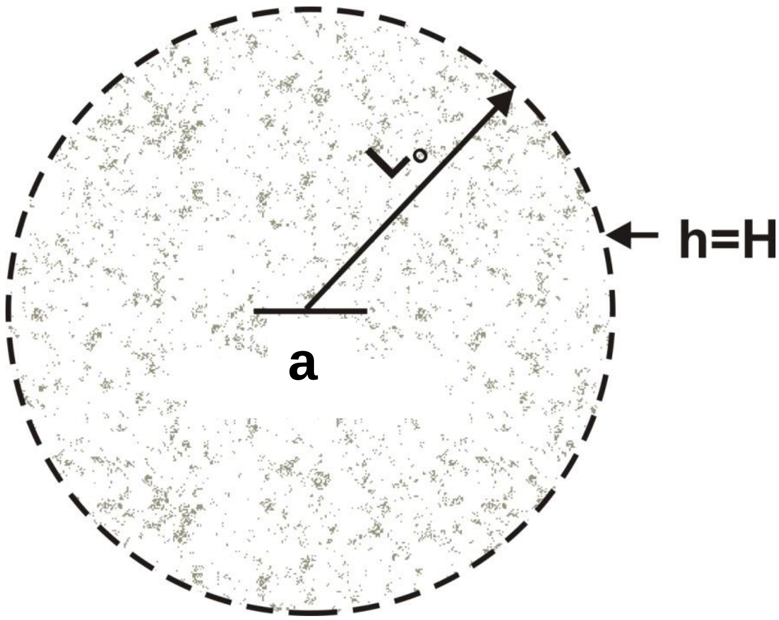


Figure 13. “Near-radial” flow for a “distant” boundary

The solution for purely radial flow is:

$$Q = 2\pi KD \frac{(H - h_{ex})}{\ln\left\{\frac{L_o}{r_e}\right\}}$$

Inflow rates are calculated with this equation and three expressions for the effective radius r_e. The results are plotted in Figure 14.

For a/b = 50, the value of r_e based on an equal area is:

$$r_{e\_A} = \sqrt{\frac{a \cdot \left(\frac{a}{50}\right)}{\pi}} = \frac{a}{12.533} = 0.0798\ a$$

$$\frac{L_0}{r_{e\_A}} = 12.533\ \frac{L_0}{a}$$

The results calculated with this value of r_e are labelled *R1.*

For a/b = 50, the value of r_e based on an equal perimeter is:

$$r_{e\_P} = \frac{a + \left(\frac{a}{50}\right)}{\pi} = 0.325\ a$$

$$\frac{L_0}{r_{e\_P}} = 3.080\ \frac{L_0}{a}$$

The results calculated with this value of r_e are labelled *R2.*

For a/b = 50, the value of r_e based on the formula of Mansur and Kaufman (1962) is:

$$r_{e\_MK} = \frac{2}{\pi}\sqrt{a \cdot \left(\frac{a}{50}\right)} = 0.090\ a$$

$$\frac{L_0}{r_{e\_MK}} = 11.107\ \frac{L_0}{a}$$

The results calculated with this value of r_e are labelled *R3.*

The results plotted in Figure 14 suggest that the asymptotic trends of Powrie and Preene's shape factors are consistent with the general trends of the calculations with the simplified radial flow analyses. The closest match is obtained with the equivalent radius of the excavation estimated on the basis of the 'equal perimeter' approach.

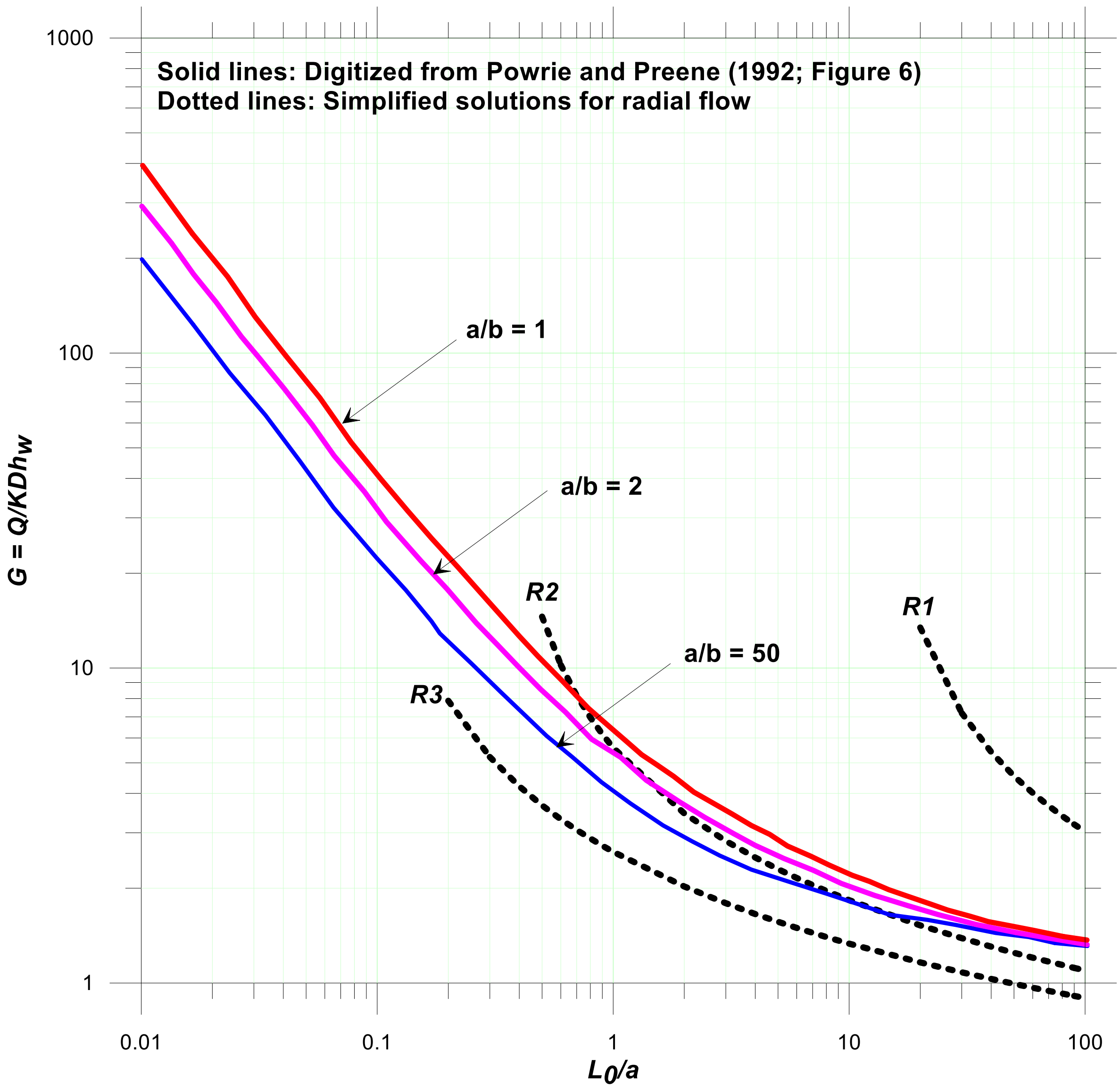


Figure 14. Assessment of the asymptotic case of a "distant" boundary

5. References

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Powers, J.P., A.B. Corwin, P.C. Schmall, and W.E. Kaeck, 2007: **Construction Dewatering and Groundwater Control**, 3rd edition, John Wiley & Sons, Inc., Hoboken, New Jersey.

Powrie, W., and M. Preene, 1992: Equivalent well analysis of construction dewatering systems, *Géotechnique*, vol. 42, no. 4, pp. 635-639.